DENSITY BOUNDS FOR SOLUTIONS TO DIFFERENTIAL EQUATIONS DRIVEN BY GAUSSIAN ROUGH PATHS

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Abstract. We consider finite dimensional rough differential equations driven by centered Gaussian processes. Combining Malliavin calculus, rough paths techniques and interpolation inequalities, we establish upper bounds on the density of the corresponding solution for any fixed time $t > 0$. In addition, we provide Varadhan estimates for the asymptotic behavior of the density for small noise. The emphasis is on working with general Gaussian processes with covariance function satisfying suitable abstract, checkable conditions.

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1. Introduction

Let p_t be the density of the solution Y_t^z to a stochastic differential equation

$$Y_t^z = z + \int_0^t V_0(Y_s^z)ds + \sum_{i=1}^d \int_0^t V_i(Y_s^z)dB_s^i, \tag{1}$$

driven by a d-dimensional Brownian motion B, where $z \in \mathbb{R}^n$ is a given initial condition and $V_0, \ldots, V_d$ are smooth vector fields on $\mathbb{R}^n$. In this classical setting and under non-degeneracy conditions on the vector fields $V_0, \ldots, V_d$, it is a well-know fact that p_t behaves like a Gaussian density. Such results can be obtained by considering the PDE governing p_t, which relies on the Markovian nature of (1). Alternatively, due to the celebrated proof of Hörmander's theorem by Malliavin [25], more probabilistic tools have been used in order to analyze laws of solutions to stochastic differential equations. This kind of technology has paved the way to the extension of such results to a much broader class of differential equations, such as delayed equations [6, 14] and stochastic PDE (see e.g [1, 28, 30] among many others).

While the above equation (1) is restricted to Brownian noise, Terry Lyons' theory of rough paths allows to study more general stochastic differential equations of the type

$$Z_t^z = z + \int_0^t V_0(Z_s^z)ds + \sum_{i=1}^d \int_0^t V_i(Z_s^z)dX_s^i, \tag{2}$$

driven by general p-rough paths X. Among the processes X to which the abstract theory of rough paths can be applied, fractional Brownian motion has attracted a lot of attention in recent years. Indeed, based on several recent works in this direction, the law of the solution to (2) driven by fractional Brownian motion is now fairly well understood. Important results in this direction include the existence of a density, smoothness results, Gaussian bounds, short time asymptotics, invariant measures, hitting probabilities and the existence of local times (see [2, 8, 10, 5, 4, 22, 19, 3, 24] and the references therein).

Much less is known for differential equations (2) driven by general Gaussian processes. This is in contrast to the theory of rough paths, which covers a lot more than fractional Brownian motion. In fact, the existence of a rough path lift for Gaussian processes is naturally related to the existence of 2-d Young type integrals for the covariance function R, as highlighted in [17] and improved in [12] based on mixed variations of R. In addition, in [12] the applicability to a wide variety of Gaussian processes, such as Gaussian random Fourier series and bifractional Brownian motions is shown, hence allowing to give a meaning and solve equations of the form (2) in this general framework. Further studies of differential equations driven by general Gaussian processes include Hörmander type theorems under general local non-determinism type conditions on the covariance R (see [10]).

The current article is a further development towards a more complete description of differential equations (2) driven by general Gaussian processes. More precisely, we consider (2) driven by a Gaussian process X satisfying appropriate general, checkable conditions. Assuming ellipticity conditions on the vector fields $V_0, \ldots, V_d$ and natural conditions on the covariance R, we prove that the density of Z_t admits a sub Gaussian upper bound (Theorem 3.4 below). Moreover, we show in Theorem 4.7 below that the density satisfies Varadhan type estimates for small noise. The proof of the above results is based on stochastic analysis tools and, more specifically, on an integration by parts formula which gives an exact expression for the density function in terms of the Malliavin derivatives and the Malliavin matrix of Z. Thus, a large part of the paper is devoted to obtaining precise estimates for the Malliavin derivative and Malliavin matrix.

The assumptions on the driving Gaussian process are quite standard in the rough paths literature and can be divided into the following two groups:

(i) Similarly to [12], we assume that the covariance function R has finite mixed $(1,\rho)$-variation for some $\rho \in [1,2)$ in order to ensure that the driving process X admits a rough path lift and complementary Young regularity is satisfied.

(ii) In order to analyze the inverse of the Malliavin matrix of the solution Z, we rely on interpolation inequalities for the Cameron-Martin space related to X (see Proposition 2.23 below), which in turn rely on monotonicity conditions on the increments of the covariance R (see Hypotheses 2.18 below) and so-called non-determinism conditions (Hypothesis 2.21 below), which have already been used in [10].

The rest of the paper is organized as follows. In Section 2, we provide some basic tools from Malliavin calculus and rough path theory that will be needed later. We also set up corresponding notations in this section. Section 3 is devoted to obtaining the upper bound of the density, while Section 4 focuses on Varadhan estimates. Finally, in Section 5, we provide several examples of Gaussian rough paths that satisfy the general assumptions supposed in the main body of this work.

Notations: Throughout this paper, unless specified otherwise, we denote Euclidean norms by $|\cdot|$. The space of $\mathbb{R}^n$-valued γ-Hölder continuous functions defined on $[0,T]$ will be denoted by $\mathcal{C}^\gamma([0,T],\mathbb{R}^n)$ and $\mathcal{C}^\gamma$ for short. For a function $g \in \mathcal{C}^\gamma([0,T],\mathbb{R}^n)$ and $0 \le s < t \le T$, we shall consider the semi-norms

$$\|g\|_{\gamma;[s,t]} := \sup_{s \le u < v \le t} \frac{|g_v - g_u|}{|v-u|^\gamma}. \tag{3}$$

Generic universal constants will be denoted by c, C independently of their exact values.

2. Preliminary material

This section contains some basic tools from Malliavin calculus and rough paths theory, as well as some analytical results, which are crucial for the definition and analysis of equation (2).

2.1. **Rough path above X.**

In this section we shall recall the notion of a rough path above a signal x, and how this applies to Gaussian signals. The interested reader is referred to [15, 17, 18] for further details.

For $s < t$ and $m \ge 1$, consider the simplex $\Delta^m_{st} = \{(u_1, \ldots, u_m) \in [s,t]^m;\, u_1 < \cdots < u_m\}$, while the simplices over $[0,T]$ will be denoted by Δ^m. The definition of a rough path above a signal x relies on the following notion of increments.

Definition 2.1. *Let $k \geq 1$. Then the space of $(k-1)$-increments, denoted by $\mathcal{C}_k([0,T],\mathbb{R}^n)$ or simply $\mathcal{C}_k(\mathbb{R}^n)$, is defined as*

$$\mathcal{C}_k(\mathbb{R}^n) \equiv \left\{ g \in C(\Delta^k;\mathbb{R}^n);\ \lim_{t_i \to t_{i+1}} g_{t_1 \cdots t_k} = 0,\ i \leq k-1 \right\}.$$

We now introduce a finite difference operator called δ, which acts on increments and is useful to split iterated integrals into simpler pieces.

Definition 2.2. *Let $g \in \mathcal{C}_1(\mathbb{R}^n)$, $h \in \mathcal{C}_2(\mathbb{R}^n)$. Then for $(s,u,t) \in \Delta^3$, we set*

$$\delta g_{st} = g_t - g_s, \qquad \text{and} \qquad \delta h_{sut} = h_{st} - h_{su} - h_{ut}.$$

The regularity of increments in $\mathcal{C}_2$ will be measured in terms of p-variation as follows.

Definition 2.3. *For $f \in \mathcal{C}_2(\mathbb{R}^n)$, $p > 0$ we set*

$$\|f\|_{p-\mathrm{var}} = \|f\|_{p-\mathrm{var};[0,T]} = \sup_{\Pi \subset [0,T]} \left(\sum_i |f_{t_i t_{i+1}}|^p \right)^{1/p},$$

where the supremum is taken over all subdivisions Π of $[0,T]$. The set of increments in $\mathcal{C}_2(\mathbb{R}^n)$ with finite p-variation is denoted by $\mathcal{C}_2^{p-\mathrm{var}}(\mathbb{R}^n)$.

With these preliminary definitions at hand, we can now define the notion of a rough path.

Definition 2.4. *Let x be a continuous $\mathbb{R}^d$-valued path with finite p-variation for some $p \geq 1$. We say that x gives rise to a geometric p-rough path if there exist*

$$\left\{ \mathbf{x}_{st}^{\mathbf{n};i_1,\ldots,i_n};\ (s,t) \in \Delta^2,\ n \leq \lfloor p \rfloor,\ i_1,\ldots,i_n \in \{1,\ldots,d\} \right\},$$

such that $\mathbf{x}_{st}^{\mathbf{1}} = \delta x_{st}$ and

(1) Regularity: *For all $n \leq \lfloor p \rfloor$, each component of $\mathbf{x}^{\mathbf{n}}$ has finite $\frac{p}{n}$-variation in the sense of Definition 2.3.*

(2) Multiplicativity: *With $\delta \mathbf{x}^{\mathbf{n}}$ as in Definition 2.2 we have*

$$\delta \mathbf{x}_{sut}^{\mathbf{n};i_1,\ldots,i_n} = \sum_{n_1=1}^{n-1} \mathbf{x}_{su}^{\mathbf{n}_1;i_1,\ldots,i_{n_1}} \mathbf{x}_{ut}^{\mathbf{n}-\mathbf{n}_1;i_{n_1+1},\ldots,i_n}. \tag{4}$$

(3) Geometricity: *Let x^ε be a sequence of piecewise smooth approximations of x. For any $n \leq \lfloor p \rfloor$ and any set of indices $i_1,\ldots,i_n \in \{1,\ldots,d\}$, we assume that $\mathbf{x}^{\varepsilon,\mathbf{n};i_1,\ldots,i_n}$ converges in $\frac{p}{n}$-variation to $\mathbf{x}^{\mathbf{n};i_1,\ldots,i_n}$, where $\mathbf{x}_{st}^{\varepsilon,\mathbf{n};i_1,\ldots,i_n}$ is defined for $(s,t) \in \Delta_2$ by*

$$\mathbf{x}_{st}^{\varepsilon,\mathbf{n};i_1,\ldots,i_n} = \int_{(u_1,\ldots,u_n) \in \Delta_{st}^n} dx_{u_1}^{\varepsilon,i_1} \cdots dx_{u_n}^{\varepsilon,i_n}.$$

Hypothesis 2.5. *Let x be a continuous $\mathbb{R}^d$-valued path with finite p-variation for $p \geq 1$. We assume that x gives rise to a geometric rough path in the sense of Definition 2.4.*

We can now state the main theorem concerning the existence and uniqueness of the solution to a rough differential equation. We refer the reader to [15, 18] for its proof.

Theorem 2.6. *Let X be a geometric p-rough path and $V_0, \ldots, V_d$ be $\mathcal{C}^\gamma$-Lipschitz continuous vector fields in $\mathbb{R}^n$ for some $\gamma > p \geq 1$. For $\varepsilon > 0$, let Z^ε be the unique solution of the following ordinary differential equation on $[0, T]$*

$$Z_t^\varepsilon = z + \int_0^t V_0(Z_s^\varepsilon) ds + \sum_{i=1}^d \int_0^t V_i(Z_s^\varepsilon) dX_s^{\varepsilon,i}, \tag{5}$$

where X^ε is a piecewise linear approximation of X as in Definition 2.4. Then Z^ε converges in p-variation to a path Z, which can be seen as the unique solution of equation (2) understood in rough path sense.

Now we assume that $X_t = (X_t^1, ..., X_t^d)$ is a continuous, centered Gaussian process with i.i.d. components, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The covariance function of X, is defined as follows

$$R(s,t) := \mathbb{E}\left[X_s^j X_t^j\right], \tag{6}$$

where X^j is any of the components of X. We shall also use the following notation in the sequel

$$\sigma_t^2 := \mathbb{E}\left[\left(X_t^j\right)^2\right], \quad \text{and} \quad \sigma_{s,t}^2 := \mathbb{E}\left[\left(\delta X_{st}^j\right)^2\right]. \tag{7}$$

A lot of the information concerning X is encoded in the rectangular increments of the covariance function R, which is given by

$$R_{uv}^{st} := \mathbb{E}\left[\left(X_t^j - X_s^j\right)\left(X_v^j - X_u^j\right)\right]. \tag{8}$$

The 2D ρ-variation of R on a rectangle $[0,t]^2$ is given by

$$V_\rho(R; [0,t]^2) := \sup\left\{\left(\sum_{i,j}\left|R_{s_i s_{i+1}}^{t_j t_{j+1}}\right|^\rho\right)^{1/\rho}; \, (s_i), (t_j) \in \Pi\right\}, \tag{9}$$

where Π is the set of partitions of $[0, T]$. For simplicity, we denote $V_\rho(R) = V_\rho(R; [0,T]^2)$ in the following. The following result (borrowed from [17]) relates the ρ-variation of R with the pathwise assumptions allowing to apply the abstract rough paths theory.

Proposition 2.7. *Let $X = (X^1, \ldots, X^d)$ be a continuous, centered Gaussian process with i.i.d. components and covariance function R defined by (6). If R has finite 2D ρ-variation for some $\rho \in [1, 2)$, then X satisfies Hypothesis 2.5, provided $p > 2\rho$.*

As a direct application of Theorem 2.6 and Proposition 2.7, we notice that whenever a Gaussian process X admits a covariance function R with finite $2D$ ρ-variation (and $\rho \in [1, 2)$), then equation (2) driven by X admits a unique solution in the rough path sense. In the sequel we shall give some information about the law of this solution Z.

2.2. **Wiener space associated to general Gaussian processes.** In this section we consider again the continuous, centered Gaussian process X of Section 2.1. Recall that its covariance function R is defined by (6). Our analysis is based on two different (though related) Hilbert spaces $\mathcal{H}$, $\bar{\mathcal{H}}$. Roughly speaking, the space $\bar{\mathcal{H}}$ is the usual Cameron-Martin (or reproducing kernel Hilbert) space of X, while $\mathcal{H}$ is the space allowing a proper definition of Wiener integrals as defined e.g in [27].

The Cameron-Martin space $\bar{\mathcal{H}}$ is defined to be the completion of the linear space of functions of the form

$$\bar{\mathcal{E}} = \left\{ \sum_{i=1}^{n} a_i R(t_i, \cdot), \quad a_i \in \mathbb{R} \text{ and } t_i \in [0,T] \right\},$$

with respect to the following inner product

$$\left\langle \sum_{i=1}^{n} a_i R(t_i, \cdot), \sum_{j=1}^{m} b_j R(s_j, \cdot) \right\rangle_{\bar{\mathcal{H}}} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j R(t_i, s_j). \tag{10}$$

The space $\mathcal{H}$ is defined similarly, but this time we are considering the completion of the set of step functions

$$\mathcal{E} = \left\{ \sum_{i=1}^{n} a_i \mathbf{1}_{[0,t_i]} : a_i \in \mathbb{R},\, t_i \in [0,T] \right\},$$

with respect to the inner product

$$\left\langle \sum_{i=1}^{n} a_i \mathbf{1}_{[0,t_i]}, \sum_{j=1}^{m} b_j \mathbf{1}_{[0,s_j]} \right\rangle_{\mathcal{H}} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j R(t_i, s_j). \tag{11}$$

Remark 2.8. Let $X_0 = 0$ and thus $R(0,0) = 0$. Then, as suggested by (11), for any $h_1, h_2 \in \mathcal{H}$, we have

$$\langle h_1, h_2 \rangle_{\mathcal{H}} = \int_0^T \int_0^T h_1(s) h_2(t) dR(s,t), \tag{12}$$

whenever the 2D Young's integral on the right-hand side is well-defined (see, e.g., [9, Proposition 4] for details).

Since $\mathcal{H}$ is the completion of $\mathcal{E}$ w.r.t $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, it is obvious that the linear map $\mathcal{R} : \mathcal{E} \to \bar{\mathcal{H}}$ defined by

$$\mathcal{R}\left(\mathbf{1}_{[0,t]}\right) = R(t, \cdot) \tag{13}$$

extends to an isometry between $\mathcal{H}$ and $\bar{\mathcal{H}}$. We also recall that $\mathcal{H}$ is isometric to the Hilbert space $H^1(Z) \subseteq L^2(\Omega, \mathcal{F}, \mathbb{P})$ which is defined to be the $|\cdot|_{L^2(\Omega)}$-closure of the set

$$\left\{ \sum_{i=1}^{n} a_i X_{t_i} : a_i \in \mathbb{R},\ t_i \in [0,T],\ n \in \mathbb{N} \right\}.$$

In particular, we have that $\left|\mathbf{1}_{[0,t]}\right|_{\mathcal{H}} = |X_t|_{L^2(\Omega)}$. The isometry generated by (13) is denoted by $X(\phi)$, and is called Wiener integral.

Remark 2.9. Since the space $\mathcal{H}$ is a closure of indicator functions, it is easily defined on any interval $[a,b] \subset [0,T]$. We denote by $\mathcal{H}([a,b])$ this restriction. For $[a,b] \subset [0,T]$, one can then check the following identity by a limiting procedure on simple functions

$$\left\langle f\, \mathbf{1}_{[a,b]},\, g\, \mathbf{1}_{[a,b]} \right\rangle_{\mathcal{H}} = \langle f,\, g \rangle_{\mathcal{H}([a,b])}. \tag{14}$$

The rough path analysis of Gaussian processes relies heavily on embedding results for the Cameron-Martin space $\bar{\mathcal{H}}$ into spaces of functions of finites p-variation. In the following we shalll recall a recent embedding result from [12]. To this aim, let us recall the definition of the mixed (γ, ρ)-variation given in [32].

Definition 2.10. *For a general continuous function $R : [0,T]^2 \to \mathbb{R}$ and two parameters $\gamma, \rho \geq 1$, we set*

$$V_{\gamma,\rho}(R;[s,t]\times[u,v]) := \sup_{\substack{(t_i)\in\mathcal{D}([s,t])\\ (t'_j)\in\mathcal{D}([u,v])}} \left(\sum_{t'_j} \left(\sum_{t_i} \left| R^{t'_j t'_{j+1}}_{t_i t_{i+1}} \right|^{\gamma} \right)^{\frac{\rho}{\gamma}} \right)^{\frac{1}{\rho}}, \tag{15}$$

where $\mathcal{D}([s,t])$ denotes the set of all dissections of $[s,t]$ and where we have set

$$R^{t'_j t'_{j+1}}_{t_i t_{i+1}} = R(t_{i+1}, t'_{j+1}) - R(t_{i+1}, t'_j) - R(t_i, t'_{j+1}) + R(t_i, t'_j).$$

Observe that, whenever the function R in Definition 2.10 is given as a covariance function as in (6), then the rectangular increment $R^{t'_j t'_{j+1}}_{t_i t_{i+1}}$ is given by (8). In addition, the ρ-variation of R introduced in (9) and invoked in Proposition 2.7 is recovered as $V_\rho = V_{\rho,\rho}$. As a last elementary remark, also notice that

$$V_{\gamma\vee\rho}(R;A) \leq V_{\gamma,\rho}(R;A) \leq V_{\gamma\wedge\rho}(R;A),$$

for all rectangles $A \subseteq [0,T]^2$. We set, for future use

$$\kappa^2_{s,t} := V_{1,\rho}(R;[s,t]^2), \quad \text{and} \quad \kappa^2_t := V_{1,\rho}(R;[0,t]^2). \tag{16}$$

With these elementary notions at hand, we next introduce an hypothesis which allows the use of both rough paths techniques and tools from stochastic analysis for the underlying process.

Hypothesis 2.11. *Let X be a d-dimensional continuous, centered Gaussian process with i.i.d. components and covariance R defined by (6). We assume that the function R admits a finite mixed $(1,\rho)$-variation, as introduced in Definition 2.10, for some $\rho \in [1,2)$.*

Remark 2.12. Since the mixed $(1,\rho)$-variation of R controls $V_\rho(R)$, Proposition 2.7 and Hypothesis 2.11 imply the existence of a rough path lift of X to a p-variation rough path with $p > 2\rho$.

Definition 2.13. *Given $\rho \in [1,2)$, we say that R has finite Hölder-controlled mixed $(1,\rho)$-variation if there exists a $C > 0$ such that for all $0 \leq s \leq t \leq T$ we have*

$$V_{1,\rho}(R;[s,t]^2) \leq C(t-s)^{1/\rho}.$$

Remark 2.14. An important consequence of R having finite Hölder-controlled mixed $(1,\rho)$-variation is that $\mathbf{X}$ has $1/p$-Hölder continuous sample paths for every $p > 2\rho$. This will be needed in order to obtain the interpolation inequality in Proposition 2.23 below which plays an important role in the analysis.

Remark 2.15. Similarly to the argument in [10, Remark 2.4], for any process X satisfying Hypothesis 2.11, one can introduce a deterministic time-change $\tau : [0,T] \to [0,T]$ such that $\tilde{X} = X \circ \tau$ has finite Hölder-controlled mixed $(1,\rho)$-variation.

We are now ready to recall an embedding result for the Cameron-Martin space $\bar{\mathcal{H}}$, obtained in [12].

Theorem 2.16. *Let X be a centered Gaussian process satisfying Hypothesis 2.11 and recall that $\bar{\mathcal{H}}$ is defined by the inner product (10). Then there is a continuous embedding*

$$\bar{\mathcal{H}} \hookrightarrow C^{q-\mathrm{var}}, \qquad \text{with} \qquad q = \frac{1}{\frac{1}{2\rho} + \frac{1}{2}} < 2.$$

More precisely, the following inequality holds true

$$\|h\|_{q-\mathrm{var};[s,t]} \leq \kappa_{s,t} \, \|h\|_{\bar{\mathcal{H}}}, \quad \forall [s,t] \subseteq [0,T],$$

where the constant $\kappa_{s,t}$ is defined by (16).

Finally we can give a statement which will be the basis of the interpretation of several integrals related to Malliavin derivatives

Corollary 2.17. *Let X be a centered Gaussian process satisfying Hypothesis 2.11 for a given $\rho \in [1,2)$, let $\bar{\mathcal{H}}$ be the Cameron-Martin space related to X and let $\varepsilon \in (0, 2-\rho]$ small enough. Then*

(i) *The process X gives rise to a finite p-variation rough path for $p = 2\rho + \varepsilon$.*

(ii) *The spaces $\bar{\mathcal{H}}$ and $\mathcal{C}^{p-\mathrm{var}}$ satisfy Young's complementary condition. Namely, there exists a q such that $\bar{\mathcal{H}}$ is embedded in $\mathcal{C}^{q-\mathrm{var}}$ and such that $p^{-1} + q^{-1} > 1$.*

Proof. Item (i) follows from Remark 2.12. As far as item (ii) is concerned, we invoke Theorem 2.16 and we take $q = (\frac{1}{2\rho} + \frac{1}{2})^{-1}$. Since $\rho < 2$ and since we have chosen $p = 2\rho + \varepsilon$ with ε small enough, it is easily checked that $p^{-1} + q^{-1} > 1$.

□

2.3. **Interpolation inequalities.** Interpolation inequalities involving Cameron-Martin spaces are crucial in order to bound Malliavin derivatives which appear in density formulae. In this section we derive such inequalities for a general Gaussian process, under conditions introduced in [10, 12]. The first condition we shall impose concerns correlations of increments.

Hypothesis 2.18. *Let X be an $\mathbb{R}^d$-valued centered Gaussian process X with i.i.d. coordinates and covariance function R. In the sequel we assume that:*

(i) *X has non-positively correlated increments, that is, for all $(t_1, t_2, t_3, t_4) \in \Delta^4$ and every coordinate $j = 1, \ldots, d$ we have*

$$R_{t_1 t_2}^{t_3 t_4} = \mathbb{E}\left[\delta X_{t_1 t_2}^j \, \delta X_{t_3 t_4}^j\right] \leq 0. \tag{17}$$

(ii) *The covariance R is diagonally dominant. That is, for all $(t_1, t_2, t_3, t_4) \in \Delta^4$ and every coordinate $j = 1, \ldots, d$ we have*

$$R_{t_2 t_3}^{t_1 t_4} = \mathbb{E}\left[\delta X_{t_2 t_3}^j \, \delta X_{t_1 t_4}^j\right] \geq 0. \tag{18}$$

With this Hypothesis at hand, we start with some inequalities which stem from the Cameron-Martin embedding Theorem 2.16.

Proposition 2.19. *Let X be a continuous, centered Gaussian process starting from zero, with i.i.d. components and covariance function R satisfying Hypothesis 2.11. Further, let $q = (\frac{1}{2\rho} + \frac{1}{2})^{-1}$ and consider $p \geq 1$ such that $1/p + 1/q > 1$. Then*

(i) *There exist constants $c_1, c_2 > 0$ such that for every $f \in \mathcal{H}$ and $t \in (0,T]$, we have*

$$\|f\mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2 \le c_2\, \kappa_t^2 \left(\|f\mathbf{1}_{[0,t]}\|_{p-var}^2 + \|f\mathbf{1}_{[0,t]}\|_{\infty}^2 \right),$$

where κ_t is as in (16).

(ii) *Assume that X satisfies Hypothesis 2.18 and let $\mathcal{C}^\gamma$ be the space of γ-Hölder continuous functions. Then, for any continuous $f \in \mathcal{H} \cap \mathcal{C}^\gamma$ with $1/\rho + \gamma > 1$,*

$$\|f\mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2 \ge \int_0^t f^2(r) R(dr, t) \ge \sigma_t^2 \min_{[0,t]} |f|^2, \tag{19}$$

where σ_t^2 is as in (7).

Remark 2.20. Equation (19) above is in fact a consequence of [10, Proposition 6.6], by taking $s = 0$ and $t = T$ therein. We have included a more elementary proof here for sake of clarity.

Proof of Proposition 2.19. We prove the two items of this proposition separately.

Proof of (i). Recall that the spaces $\mathcal{H}([a,b])$ are introduced in Remark 2.9. As mentioned in [29], the following relation holds true for any $h_1, h_2 \in \mathcal{H}([0,t])$

$$\langle h_1, h_2 \rangle_{\mathcal{H}([0,t])} = \int_0^t h_1 d\mathcal{R} h_2,$$

where the right hand side is understood in the Young sense and $\mathcal{R}$ is the isometry going from $\mathcal{H}([0,t])$ to $\bar{\mathcal{H}}([0,t])$, as given in relation (13). Hence, if $p^{-1} + q^{-1} > 1$, classical inequalities for Young's integral imply

$$|\langle h_1, h_2 \rangle_{\mathcal{H}([0,t])}| \le C(\|h_1\|_{p-var;[0,t]} + \|h_1\|_{\infty;[0,t]}) \|\mathcal{R}h_2\|_{q-var;[0,t]}. \tag{20}$$

We now use Theorem 2.16 to get the bound

$$\|\mathcal{R}h_2\|_{q-var;[0,t]} \le \kappa_t \, \|\mathcal{R}h_2\|_{\bar{\mathcal{H}}([0,t])} = \kappa_t \, \|h_2\|_{\mathcal{H}([0,t])},$$

where we recall that we have set $\kappa_t^2 = V_{1,\rho}(R; [0,t]^2)$. Plugging this information back into (20) and choosing $h_1 = h_2 = f$, we obtain

$$\begin{aligned} \|f\|_{\mathcal{H}([0,t])}^2 &= |\langle f, f \rangle_{\mathcal{H}([0,t])}| \le C(\|f\|_{p-var;[0,t]} + \|f\|_{\infty;[0,t]}) \|\mathcal{R}f\|_{q-var;[0,t]} \\ &\le C\kappa_t(\|f\|_{p-var;[0,t]} + \|f\|_{\infty;[0,t]}) \|f\|_{\mathcal{H}([0,t])}. \end{aligned}$$

Dividing this expression by $\|f\|_{\mathcal{H}([0,t])}$ finishes the proof of claim (i).

Proof of (ii). We first prove the claim for elementary step functions. Namely, consider $t \le T$, a partition (t_i) of the interval $[0,t]$, and set

$$f\mathbf{1}_{[0,t]} = \sum_i a_i \mathbf{1}_{[t_i, t_{i+1}]}.$$

Then the following identity obviously holds true

$$\|f\mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2 = \sum_{i,j} a_i a_j \left\langle \mathbf{1}_{[t_i,t_{i+1}]}, \mathbf{1}_{[t_j,t_{j+1}]} \right\rangle_{\mathcal{H}} = \sum_{i,j} a_i a_j R_{t_j,t_{j+1}}^{t_i,t_{i+1}}.$$

We now separate diagonal and non-diagonal terms in order to get

$$\|f\mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2 = \sum_i \sum_{j \ne i} a_i a_j R_{t_j,t_{j+1}}^{t_i,t_{i+1}} + \sum_i a_i^2 R_{t_i,t_{i+1}}^{t_i,t_{i+1}} \ge S_1 - S_2, \tag{21}$$

where S_1 and S_2 are defined by

$$S_1 = \sum_i a_i^2 R^{t_i,t_{i+1}}_{t_i,t_{i+1}}, \quad \text{and} \quad S_2 = \sum_i \sum_{j\neq i} |a_i||a_j| \left| R^{t_i,t_{i+1}}_{t_j,t_{j+1}} \right|.$$

Next, in order to bound S_2 from above, we first invoke the elementary inequality $2|a_i||a_j| \leq |a_i|^2 + |a_j|^2$ to get

$$S_2 \leq \frac{1}{2} \sum_i \sum_{j\neq i} a_i^2 \left| R^{t_i,t_{i+1}}_{t_j,t_{j+1}} \right| + \frac{1}{2} \sum_i \sum_{j\neq i} a_j^2 \left| R^{t_i,t_{i+1}}_{t_j,t_{j+1}} \right|.$$

Then, using (17), we get

$$S_2 \leq -\frac{1}{2} \sum_i \sum_{j\neq i} a_i^2 R^{t_i,t_{i+1}}_{t_j,t_{j+1}} - \frac{1}{2} \sum_i \sum_{j\neq i} a_j^2 R^{t_i,t_{i+1}}_{t_j,t_{j+1}} = -\sum_i \sum_{j\neq i} a_i^2 R^{t_i,t_{i+1}}_{t_j,t_{j+1}}.$$

Inserting this in (21) yields

$$\|f\mathbf{1}_{[0,t]}\|^2_{\mathcal{H}} \geq \sum_{i,j} a_i^2 R^{t_i,t_{i+1}}_{t_j,t_{j+1}} = \sum_i a_i^2 R^{t_i t_{i+1}}_{0t}. \tag{22}$$

Let us observe that, owing to the diagonal dominance assumption (18), the measure $R(dr,t)$ defined by

$$R([u,v],t) := R^{uv}_{0t}$$

is non-negative. Furthermore, one can recast inequality (22) as

$$\|f\mathbf{1}_{[0,t]}\|^2_{\mathcal{H}} \geq \int_0^t f^2(r) R(dr,t).$$

Using elementary properties of positive measures, we thus end up with

$$\|f\mathbf{1}_{[0,t]}\|^2_{\mathcal{H}} \geq \min_{[0,t]} |f|^2 R^{0t}_{0t} = \min_{[0,t]} |f|^2 \sigma_t^2,$$

which proves the claim (ii) for elementary functions f. Finally, we show that the above remains true all $f \in \mathcal{H} \cap \mathcal{C}^\gamma$. Let $D = \{t_i : i = 0, 1, ..., n\}$ be any partition of $[0,T]$, and set $f_D(t) = f(t_i), t_i \leq t < t_{i+1}$. Since f_D is an elementary function, we have

$$\int_{[0,t]^2} f_D(s) f_D(t) dR(s,t) = \|f_D \mathbf{1}_{[0,t]}\|^2_{\mathcal{H}} \geq \min_{[0,t]} |f_D|^2 \sigma_t^2.$$

Note that we assume $f \in \mathcal{C}^\gamma$ with $1/\rho + \gamma > 1$. The left hand-side of the above display is the Riemann sum approximation to the 2D Young integral of f against R along the partition D. Hence, if we shrink the mesh of the partition D,

$$\int_{[0,t]^2} f_D(s) f_D(t) dR(s,t) \to \int_{[0,t]^2} f(s) f(t) dR(s,t) = \|f\mathbf{1}_{[0,t]}\|^2_{\mathcal{H}}.$$

On the other hand, $\min_{[0,t]} |f_D| \to \min_{[0,t]} |f|$, when shrinking the mesh of D, by the construction of f_D and the fact that f is continuous. The proof is thus completed. □

We now wish to get a non-degeneracy result for the norm in $\mathcal{H}$, that is, a lower bound on $\|f\|_{\mathcal{H}}$ involving $\|f\|_\infty$. This requires the following additional hypothesis.

Hypothesis 2.21. *Let $(X_t)_{t\in[0,T]}$ be a centered continuous $\mathbb{R}^d$-valued Gaussian process. For any $0 \le a \le b \le T$, denote by $\mathcal{F}_{a,b}$ the following σ-algebra*

$$\mathcal{F}_{a,b} = \sigma(\delta X_{uv} : a \le u \le v \le b).$$

Then we assume that there exists an $\alpha > 0$ such that

$$\inf_{0\le s<t\le T} \frac{1}{(t-s)^\alpha} \mathrm{Var}\left(\delta X_{st} | \mathcal{F}_{0,s} \vee \mathcal{F}_{t,T}\right) = c_X > 0. \tag{23}$$

We call the smallest α that satisfies the above condition the index of non-determinism.

Remark 2.22. Note that since we are working with Gaussian processes, the above conditional variance $\mathrm{Var}\left(\delta X_{st} | \mathcal{F}_{0,s} \vee \mathcal{F}_{t,T}\right)$ is deterministic. Moreover, assuming Hypothesis 2.21 holds true and setting $s = 0$ in (23), the law of total variance gives us

$$\sigma_t^2 = \mathrm{Var}\left(X_t\right) \ge \mathrm{Var}\left(\delta X_{0t} | \mathcal{F}_{0,0} \vee \mathcal{F}_{t,T}\right) \ge c_X t^\alpha,$$

with σ_t^2 as in (7).

With Hypothesis 2.21 at hand, we borrow the following interpolation inequality from [10, Corollary 6.10].

Proposition 2.23. *Let $(X_t)_{t\in[0,T]}$ be a continuous Gaussian process starting from zero with covariance function $R : [0,T]^2 \to \mathbb{R}$. Suppose Hypothesis 2.18 and 2.21 are satisfied. Furthermore, we assume that R has finite Hölder-controlled mixed $(1,\rho)$-variation for some $\rho \in [1,2)$ in the sense of Definition 2.13. Then there exists a universal constant c such that for any $f \in C^\gamma([0,T],\mathbb{R})$ with $\gamma + 1/\rho > 1$, we have*

$$\|f\|_{\infty;[0,T]} \le 2 \max\left\{ \frac{\|f\|_{\mathcal{H}}}{\sigma_T},\ \frac{1}{\sqrt{c_X}} \|f\|_{\mathcal{H}}^{\frac{2\gamma}{2\gamma+\alpha}} \|f\|_{\gamma;[0,T]}^{\frac{\alpha}{2\gamma+\alpha}} \right\}, \tag{24}$$

where c_X is the constant appearing in equation (23) and σ_t is defined by (7).

Remark 2.24. In [10], relation (24) is proved under the following additional hypothesis

$$\mathrm{Cov}(X_{s,t} X_{u,v} | \mathcal{F}_{0,s} \vee \mathcal{F}_{t,S}) \ge 0, \tag{25}$$

for any $[u,v] \subset [s,t] \subset [0,S] \subset [0,T]$. However, we are working here under the standing assumptions (17), (18) in Hypothesis 2.18, and it is shown in [10, Corollary 6.8] that (17) together with (18) implies (25).

Remark 2.25. Our interpolation inequality (24) also reads as

$$\|f\|_{\mathcal{H}} \ge \frac{\sigma_T \|f\|_{\infty;[0,T]}}{2} \min\left\{ 1,\ \frac{2\left(\frac{c_X}{2}\right)^{\frac{2\gamma+\alpha}{4\gamma}}}{\sigma_T} \frac{\|f\|_{\infty;[0,T]}^{\frac{\alpha}{2\gamma}}}{\|f\|_{\gamma;[0,T]}^{\frac{\alpha}{2\gamma}}} \right\}. \tag{26}$$

In fact we will use a slight generalization of (26) in the sequel. Namely, for all $t \le T$, Remark 2.9 asserts that $\|f \mathbf{1}_{[0,t]}\|_{\mathcal{H}} = \|f\|_{\mathcal{H}([0,t])}$. We thus get the following interpolation inequality

$$\|f \mathbf{1}_{[0,t]}\|_{\mathcal{H}} \ge \frac{\sigma_t \|f\|_{\infty;[0,t]}}{2} \min\left\{ 1,\ \frac{2\left(\frac{c_X}{2}\right)^{\frac{2\gamma+\alpha}{4\gamma}}}{\sigma_t} \frac{\|f\|_{\infty;[0,t]}^{\frac{\alpha}{2\gamma}}}{\|f\|_{\gamma;[0,t]}^{\frac{\alpha}{2\gamma}}} \right\}. \tag{27}$$

2.4. **Malliavin calculus for Gaussian processes.** In this section we review some basic aspects of Malliavin calculus. The reader is referred to [27] for further details.

As before $X_t = (X_t^1, ..., X_t^d)$ is a continuous, centered Gaussian process with i.i.d. components, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For sake of simplicity, we assume that $\mathcal{F}$ is generated by $\{X_t;\, t \in [0,T]\}$. An $\mathcal{F}$-measurable real valued random variable F is said to be cylindrical if it can be written, for some $m \geq 1$, as

$$F = f\left(X_{t_1}, \ldots, X_{t_m}\right), \quad \text{for} \quad 0 \leq t_1 < \cdots < t_m \leq 1,$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a C_b^∞ function. The set of cylindrical random variables is denoted by $\mathcal{S}$.

The Malliavin derivative is defined as follows: for $F \in \mathcal{S}$, the derivative of F in the direction $h \in \mathcal{H}$ is given by

$$\mathbf{D}_h F = \sum_{i=1}^m \frac{\partial f}{\partial x_i}\left(X_{t_1}, \ldots, X_{t_m}\right) h_{t_i}.$$

More generally, we can introduce iterated derivatives. Namely, if $F \in \mathcal{S}$, we set

$$\mathbf{D}^k_{h_1,\ldots,h_k} F = \mathbf{D}_{h_1} \ldots \mathbf{D}_{h_k} F.$$

For any $p \geq 1$, it can be checked that the operator $\mathbf{D}^k$ is closable from $\mathcal{S}$ into $\mathbf{L}^p(\Omega; \mathcal{H}^{\otimes k})$. We denote by $\mathbb{D}^{k,p}(\mathcal{H})$ the closure of the class of cylindrical random variables with respect to the norm

$$\|F\|_{k,p} = \left(\mathbb{E}\left[|F|^p\right] + \sum_{j=1}^k \mathbb{E}\left[\left\|\mathbf{D}^j F\right\|^p_{\mathcal{H}^{\otimes j}}\right]\right)^{\frac{1}{p}},$$

and we also set $\mathbb{D}^\infty(\mathcal{H}) = \cap_{p\geq 1} \cap_{k \geq 1} \mathbb{D}^{k,p}(\mathcal{H})$. The divergence operator $\delta^\diamond$ is then defined to be the adjoint operator of $\mathbf{D}$.

Estimates of Malliavin derivatives are crucial in order to get information about densities of random variables, and Malliavin matrices as well as non-degenerate random variables will feature importantly in the sequel.

Definition 2.26. *Let $F = (F^1, \ldots, F^n)$ be a random vector whose components are in $\mathbb{D}^\infty(\mathcal{H})$. Define the Malliavin matrix of F by*

$$\gamma_F = (\langle \mathbf{D}F^i, \mathbf{D}F^j \rangle_{\mathcal{H}})_{1 \leq i,j \leq n}. \tag{28}$$

Then F is called non-degenerate if γ_F is invertible a.s. and

$$(\det \gamma_F)^{-1} \in \cap_{p \geq 1} L^p(\Omega).$$

It is a classical result that the law of a non-degenerate random vector $F = (F^1, \ldots, F^n)$ admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}^n$.

2.5. **Differential equations driven by Gaussian processes.** Recall that we consider the following kind of equation

$$Z_t^z = z + \int_0^t V_0(Z_s^z)ds + \sum_{i=1}^d \int_0^t V_i(Z_s^z) dX_s^i, \tag{29}$$

where the vector fields $V_0, \ldots, V_d$ are $\mathcal{C}_b^\infty$-vector fields on $\mathbb{R}^n$ and X is a continuous, centered Gaussian process with i.i.d. components. Throughout this section, we assume that the covariance R has finite 2D ρ-variation for some $\rho \in [1, 2)$. Hence, as mentioned in Section 2.1, Proposition 2.7 implies the existence and uniqueness of a solution to (29).

Once equation (29) is solved, the vector Z_t^z is a typical example of random variable which can be differentiated in the Malliavin sense. We shall express this Malliavin derivative in terms of the Jacobian $\mathbf{J}$ of the equation, which is defined by the relation $\mathbf{J}_t^{ij} = \partial_{z_j} Z_t^{z,i}$. Setting DV_j for the Jacobian of V_j as a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, let us recall that $\mathbf{J}$ is the unique solution to the linear equation

$$\mathbf{J}_t = \mathrm{Id}_n + \int_0^t DV_0(Z_s^z)\, \mathbf{J}_s \, ds + \sum_{j=1}^d \int_0^t DV_j(Z_s^z)\, \mathbf{J}_s \, dX_s^j. \tag{30}$$

We refer to [8, 11, 29] for the following integrability and differentiability result:

Proposition 2.27. *Let X be a continuous, centered $\mathbb{R}^d$-valued Gaussian process with i.i.d. components and covariance function R having finite 2D ρ-variation for some $\rho \in [1, 2)$. Consider the solution Z^z to (29) and suppose that the vector fields V_i are $\mathcal{C}_b^\infty$. Then*

(i) *For any $\eta \geq 1$, there exists a finite constant c_η such that the Jacobian $\mathbf{J}$ defined by (30) satisfies*

$$\mathbb{E}\left[\|\mathbf{J}\|_{p-\mathrm{var};[0,T]}^\eta \right] = c_\eta. \tag{31}$$

(ii) *For every $i = 1, \ldots, n$, $t > 0$, and $z \in \mathbb{R}^n$, we have $Z_t^{z,i} \in \mathbb{D}^\infty(\mathcal{H})$ and the Malliavin derivative of Z verifies*

$$\mathbf{D}_s^j Z_t^z = \mathbf{J}_{s,t} V_j(Z_s^z), \quad j = 1, \ldots, d, \quad 0 \leq s \leq t, \tag{32}$$

where $\mathbf{D}_s^j Z_t^{z,i}$ is the j-th component of $\mathbf{D}_s Z_t^{z,i}$, and where we have set $\mathbf{J}_{s,t} = \mathbf{J}_t \mathbf{J}_s^{-1}$.

3. Upper bounds for the density

The aim of this section is to study upper bounds for the density of the solution to equation (29). Throughout this section X is a continuous, centered Gaussian process starting at zero with i.i.d. components. In addition, we assume the following uniform ellipticity condition on the vector fields.

Hypothesis 3.1. *The vector fields $V_1, \ldots, V_d$ of equation (29) are C^∞-bounded and form a uniformly elliptic system, that is, for some $\lambda > 0$,*

$$v^* V(x) V^*(x) v \geq \lambda |v|^2, \qquad \text{for all } v, x \in \mathbb{R}^n, \tag{33}$$

where we have set $V = (V_j^i)_{i=1,\ldots,n; j=1,\ldots d}$.

We further introduce

Definition 3.2. *Let X be a centered $\mathbb{R}^d$-valued Gaussian process with covariance R. We assume that X satisfies Hypothesis 2.11. Let σ_t and κ_t be as in (7), (16). We define the self-similarity parameter η_t for $t \in (0,T]$ by*

$$\eta_t := \frac{V_{1,\rho}(R;[0,t]^2)}{R(t,t)} = \left(\frac{\kappa_t}{\sigma_t}\right)^2. \tag{34}$$

Remark 3.3. The name *self-similarity parameter* for η_t stems from the fact that η_t does not depend on t whenever the Gaussian process X is self-similar. Hence, η_t can be interpreted as quantifying the lack of self-similarity.

With these definitions at hand, we shall prove an upper bound for the density of X_t, under the ellipticity assumption (33).

Theorem 3.4. *Let X be an $\mathbb{R}^d$-valued continuous, centered Gaussian process starting at zero with i.i.d. components and covariance function R. Suppose that Hypotheses 2.11, 2.18, 2.21 and 3.1 are satisfied and let $\sigma_t, \kappa_t, \eta_t$ be as in (7), (16), (34). Let Z^z be the solution to (29) driven by the Gaussian rough path lift $\mathbf{X}$ of X. Then for all $t \in (0,T]$, the density p_t of Z_t^z satisfies*

$$p_t(y) \le \frac{c_1 \eta_t^{n(n+2)}}{\kappa_t^n} \exp\left(-\frac{|y-z|^{1+\frac{1}{\rho}}}{c_2\,\kappa_t^2}\right), \quad \text{for all } y \in \mathbb{R}^n, \tag{35}$$

for some $c_1, c_2 > 0$.

The reminder of this section is devoted to prove Theorem 3.4. Our global strategy is highlighted in Section 3.1, while the main estimates are derived in Sections 3.2, 3.3 and 3.4.

3.1. Global strategy.

Our starting point in order to get the upper bound (35) is the following integration by parts type formula. Denote by $C_p^\infty(\mathbb{R}^n)$ the space of smooth functions f such that f and all of its partial derivatives have at most polynomial growth.

Proposition 3.5. [27, Proposition 2.1.4] *Let $F = (F^1, \ldots, F^n)$ be a non-degenerate random vector as in Definition 2.26. Let $G \in \mathbb{D}^\infty$ and φ be a function in the space $C_p^\infty(\mathbb{R}^n)$. Then for any multi-index $\alpha \in \{1,2,\ldots,n\}^k$, $k \ge 1$, there exists an element $H_\alpha(F,G) \in \mathbb{D}^\infty$ such that*

$$\mathbb{E}[\partial_\alpha \varphi(F) G] = \mathbb{E}[\varphi(F) H_\alpha(F,G)],$$

Moreover, the elements $H_\alpha(F,G)$ are recursively given by

$$H_{(i)}(F,G) = \sum_{j=1}^n \delta^\diamond\left(G(\gamma_F^{-1})^{ij}\,\mathbf{D}F^j\right) \quad \text{and} \quad H_\alpha(F,G) = H_{\alpha_k}(F, H_{(\alpha_1,\ldots,\alpha_{k-1})}(F,G)), \tag{36}$$

and for $1 \le p < q < \infty$ we have

$$\|H_\alpha(F,G)\|_p \le c_{p,q} \|\gamma_F^{-1}\,\mathbf{D}F\|_{k,2^{k-1}r}^k \|G\|_{k,q}^k, \tag{37}$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

As a consequence, one has the following expression for the density of a non-degenerate random vector.

Proposition 3.6. [27, Proposition 2.1.5] *Let $F = (F^1, \ldots, F^n)$ be a non-degenerate random vector as in Definition 2.26. Then the density $p_F(y)$ of F belongs to the Schwartz space, and for any $\sigma \subset \{1, \ldots, n\}$,*

$$p_F(y) = (-1)^{n-|\sigma|}\mathbb{E}[\mathbf{1}_{\{F^i>y^i, i\in\sigma, F^i<y^i, i\notin\sigma\}} H_{(1,\ldots,n)}(F,1)], \quad \textit{for all } y \in \mathbb{R}^n.$$

According to the above relation applied to $F = Z_t^z$ and $\sigma = \{i \in \{1, \ldots, n\} : y^i \geq z^i\}$, and applying inequality (37) with $k = n, p = 2, r = q = 4$, we obtain the following general upper bound for the density p_t of Z_t^z

$$p_t(y) \leq c\,\mathbb{P}(|Z_t^z - z| \geq |y - z|)^{1/2}\, \|\gamma_t^{-1}\|_{n,2^{n+2}}^n\, \|\mathbf{D}Z_t^z\|_{n,2^{n+2}}^n, \quad \text{for all } y \in \mathbb{R}^n, \tag{38}$$

where γ_t denotes the Malliavin matrix of Z_t^z. In the remainder of the section, we shall bound separately the three terms in the right hand side of (38).

3.2. **Tail estimates.** This section is devoted to estimating $\mathbb{P}(|Z_t^z - z| \geq |y - z|)$ on the right hand side of (38). Our main result in this direction is the following proposition.

Proposition 3.7. *Let X be an $\mathbb{R}^d$-valued continuous, centered Gaussian process with i.i.d. components satisfying Hypothesis 2.11 for some $\rho \in [1, 2)$. Let $\tau \in (0, T]$, κ_τ be as in (16) and Z^z, V be as in Theorem 3.4. Then there exists a constant $c_2 > 0$ such that*

$$\mathbb{P}\left(\sup_{t\leq\tau} |Z_t^z - z| \geq y\right) \leq \exp\left(-\frac{|y - z|^{1+\frac{1}{\rho}}}{c_2\, \kappa_\tau^2}\right), \tag{39}$$

for all $y \in \mathbb{R}^n$.

Proof. According to Proposition 2.7, which can be applied since the process X fulfills Hypothesis 2.11, there is a rough path lift $\mathbf{X}$ of X as in Hypothesis 2.5. For $p > 2\rho$, define the control $\omega_{\mathbf{X},p}$ by

$$\omega_{\mathbf{X},p}(s,t) = \|\mathbf{X}\|_{p-\text{var};[s,t]}^p = \sum_{n\leq\lfloor p\rfloor} \|\mathbf{X}^{\mathbf{n}}\|_{\frac{p}{n}-\text{var};[s,t]}^{1/n}. \tag{40}$$

Then [17, Lemma 10.7] asserts that

$$\|Z^z\|_{p-\text{var};[s,t]} \leq c_V \left([\omega_{\mathbf{X},p}(s,t)]^{1/p} \vee \omega_{\mathbf{X},p}(s,t) \right). \tag{41}$$

In particular, for any $t_i < t_{i+1}$ we have

$$|\delta Z_{t_i t_{i+1}}^z| \leq c_V \left([\omega_{\mathbf{X},p}(t_i, t_{i+1})]^{1/p} \vee \omega_{\mathbf{X},p}(t_i, t_{i+1}) \right). \tag{42}$$

Consider now $\alpha \geq 1$ and construct a partition of $[0, t]$ inductively in the following way: we set $t_0 = 0$ and

$$t_{i+1} := \inf\left\{ u > t_i;\ \|\mathbf{X}\|_{p-\text{var};[t_i,u]}^p \geq \alpha \right\}. \tag{43}$$

We then set $N_{\alpha,t,p} = \sup\{n \geq 0;\ t_n < t\}$. Observe that, since we have taken $\alpha \geq 1$, inequality (42) can be read as $|\delta Z_{t_i t_{i+1}}| \leq c_V\, \omega_{\mathbf{X},p}(t_i, t_{i+1}) = c_V\, \alpha$. Hence

$$|Z_t^z - z| \leq |Z_t^z - Z_{t_{N_{\alpha,t,p}}}| + \sum_{i=0}^{N_{\alpha,t,p}-1} |\delta Z_{t_i t_{i+1}}| \leq c_V\, \alpha\, (N_{\alpha,t,p} + 1). \tag{44}$$

By [11, Theorem 6.3] we have

$$\mathbb{P}\left(N_{\alpha,t,p}+1>n\right)\lesssim \exp\left(-\frac{c_{p,q,\alpha}\, n^{\frac{2}{q}}}{\kappa_t^2}\right), \tag{45}$$

where κ_t is as in (16) and q is the exponent given in Theorem 2.16 by $\frac{1}{q}=\frac{1}{2\rho}+\frac{1}{2}$. This easily implies

$$\mathbb{P}\left(\sup_{t\le\tau}|Z_t^z-z|\ge\xi\right)\le\mathbb{P}\left(c_V\,\alpha\left(N_{\alpha,\tau,p}+1\right)>\xi\right)\lesssim\exp\left(-\frac{c_{p,q,\alpha,V}\,\xi^{1+\frac{1}{\rho}}}{\kappa_\tau^2}\right), \tag{46}$$

and thus the claim. □

3.3. **Estimate for Malliavin derivatives.** We now proceed to bound the Malliavin derivatives involved in the right hand side of (38). We summarize the results in the following proposition.

Proposition 3.8. *Under the same assumptions as in Proposition 3.7, for all $m\in\mathbb{N}$ and $p>1$ there exists a positive constant $c_{m,p}$ such that*

$$\|Z_t^z\|_{m,p}\le c_{m,p}\,\kappa_t, \tag{47}$$

where $\kappa_t=V_{1,\rho}(R;[0,t]^2)^{\frac{1}{2}}$ is as in (16).

Proof. We use a method by Inahama [21] to which we refer for more details. For simplicity, we assume $V_0=0$, and first show (47) for $m=1,2$. The case $V_0\neq 0$ is treated similarly. Recall that $\mathbf{J}$ is the Jacobian process.

Step 1: Expression for the Malliavin derivatives. Let $\hat{X}=(\hat{X}_1,\dots,\hat{X}_d)$ be an independent copy of X and consider the $2d$-dimensional Gaussian process $(X,\hat{X})$. The expectation with respect to X and $\hat{X}$ are respectively denoted by $\mathbb{E}$ and $\hat{\mathbb{E}}$. Set

$$\Xi_t^1:=\sum_{j=1}^d \mathbf{J}_t\int_0^t \mathbf{J}_s^{-1}V_j(Z_s^z)d\hat{X}_s^j,$$

and

$$\Xi_t^2:=\sum_{j=1}^d \mathbf{J}_t\int_0^t \mathbf{J}_s^{-1}\left\{D^2V_j(Z_s^z)\left(\Xi_s^1,\Xi_s^1\right)dX_s^j+2DV_j(Z_t^z)\Xi_s^1,d\hat{X}_s^j\right\}.$$

Then one can show that the following bounds hold true (for more details, see equations (2.8) and (2.9) in [21], and the discussion after them),

$$\|\mathbf{D}Z_t^z\|_{\mathcal{H}\otimes\mathbb{R}^n}\le C(\hat{\mathbb{E}}|\Xi_t^1|^2)^{1/2},$$
$$\|\mathbf{D}^2Z_t^z\|_{\mathcal{H}\otimes\mathcal{H}\otimes\mathbb{R}^n}\le C(\hat{\mathbb{E}}|\Xi_t^2|^2)^{1/2}.$$

Step 2: Bound for the first order derivative. We now estimate Ξ^1 by using general bounds taken from the theory of rough paths. Namely, let

$$M=(X,\hat{X},Z^z,\mathbf{J},\mathbf{J}^{-1}). \tag{48}$$

Then, M is a rough path obtained by solving an SDE driven by $(X, \hat{X})$. Hence, it is a p-rough path for any $p > 2\rho$, where ρ is the exponent appearing in Hypothesis 2.11. Furthermore, the integral $\int \mathbf{J}_s^{-1} V(Z_s^z) d\hat{X}_s$ is a rough integral of the type $\int f(M) d\mathbf{M}$, where f has polynomial growth. We deduce that for some $r > 0$, the following bound is verified

$$|\delta \Xi_{st}^1| \le C(1 + \|\mathbf{M}\|_{p-var,[0,T]})^r \|\mathbf{M}\|_{p-var,[s,t]}. \tag{49}$$

We now estimate $\|\mathbf{M}\|_{p-var,[s,t]}$ appearing in (49). Note that both the Jacobian $\mathbf{J}$ and its inverse $\mathbf{J}^{-1}$ satisfy a linear RDE driven by X. Hence, we have the following growth-bound (cf. [11, inequality (4.10)]),

$$\|\mathbf{J}\|_{p-\mathrm{var};[0,t]} + \|\mathbf{J}^{-1}\|_{p-\mathrm{var};[0,t]} \le C \|\mathbf{X}\|_{p-var,[0,t]} \exp\left(C N_{\alpha,t,p}\right), \tag{50}$$

where $N_{\alpha,t,p}$ is defined in [11, equation (4.7)] and has finite moments of any order. Thus, gathering (50), inequality (41), the definition (48) of M and (49), we deduce that

$$|\Xi_t^1| \le C(\|\mathbf{X}\|_{p-var,[0,t]} + \|\hat{\mathbf{X}}\|_{p-var,[0,t]}) \exp\left(C N_{\alpha,t,p}\right). \tag{51}$$

We now invoke [16, Theorem 35-(i) and Corollary 66], which asserts that

$$\big\| \|\mathbf{X}\|_{p-var,[0,t]} + \|\hat{\mathbf{X}}\|_{p-var,[0,t]} \big\|_{L^q} \le C_q \kappa_t.$$

First using Hölder's inequality in (51) and then the estimate above completes the proof of (47) for $m = 1$.

Step 3: Higher order derivatives. In the same way as in Step 2, we estimate Ξ^2 as a rough integral of the type $\int \phi(M_1) d\mathbf{M}_1$ where ϕ has polynomial growth and M_1 is the rough path

$$M_1 = (X, \hat{X}, Z^z, \mathbf{J}, \mathbf{J}^{-1}, \Xi^1)$$

Arguing as before and using all the previous estimates, we obtain a bound of the same type as (51)

$$|\Xi_t^2| \le C(\|\mathbf{X}\|_{p-var,[0,t]} + \|\hat{\mathbf{X}}\|_{p-var,[0,t]}) \exp\left(C N_{\alpha,t,p}\right).$$

This easily yields the claim (47) for the case $m = 2$. Higher order Malliavin derivatives are treated similarly by constructing processes $\Xi^m, m > 2$ inductively (see [21]). $\square$

3.4. **Estimates for the Malliavin matrix.** We next provide an estimate for the inverse of the Malliavin matrix γ_t in (38).

Proposition 3.9. *Consider the solution Z^z to (29) under the same conditions as in Theorem 3.4. For $t \in (0, T]$, let γ_t be its Malliavin matrix defined as in (28). Then, for all $m \in \mathbb{N}$ and $p > 1$ there exists a constant $c_{m,p}$ such that*

$$\|\gamma_t^{-1}\|_{m,p} \le \frac{c_{m,p}\, \eta_t^m}{\sigma_t^2}, \tag{52}$$

where σ_t, η_t are as in relations (7) and (34).

Proof. Without loss of generality, we will prove (52) for $0 < t \le 1$. We divide the proof into two steps.

Step 1: case $m = 0$. Let C_t be the matrix defined by

$$C_t = \int_0^t \int_0^t \mathbf{J}_u^{-1} V(Z_u^x) V(Z_v^x)^* (\mathbf{J}_v^{-1})^* dR(u, v).$$

By Remark 2.8 and (32), we have $\gamma_t = \mathbf{J}_t C_t \mathbf{J}_t^*$. Therefore the upper bound on $\|\gamma_t^{-1}\|_p$ can be easily deduced from the following inequality

$$y^* C_t y \geq M_t \sigma_t^2 \, |y|^2, \quad \text{for} \quad y \in \mathbb{R}^n, \tag{53}$$

where M_t is a random variable admitting negative moments of any order (see, e.g. [27, Lemma 2.3.1]). To this aim, we first notice that

$$y^* C_t y = \|f \mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2, \quad \text{with} \quad f_u := V(Z_u^z)^* (\mathbf{J}_u^{-1})^* y. \tag{54}$$

Furthermore, thanks to the interpolation inequality (27), we have

$$\|f \mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2 \geq \frac{\sigma_t^2 \|f\|_{\infty;[0,t]}^2}{4} \min \left\{ 1, \; \frac{c_X \, \|f\|_{\infty;[0,t]}^{\frac{\alpha}{\gamma}}}{\sigma_t^2 \, \|f\|_{\gamma;[0,t]}^{\frac{\alpha}{\gamma}}} \right\}. \tag{55}$$

Next observe that, due to the uniform ellipticity condition $|V(x)y|^2 \geq \lambda |y|^2$, it is readily checked that

$$|f_v|^2 \geq \lambda \, |\mathbf{J}_v^{-1} y|^2 \geq \lambda \, \|\mathbf{J}_v\|^{-2} |y|^2. \tag{56}$$

Moreover, we have $J_0 = \mathrm{Id}$, which implies that $\sup\{\|J_v\|^{-1}; v \in [0,t]\} \geq 1$. Relation (56) thus yields

$$\|f\|_{\infty;[0,t]} \geq \lambda |y|. \tag{57}$$

Plugging (57) into (55), we thus get

$$\|f \mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2 \geq \sigma_t^2 M_t |y|^2, \quad \text{with} \quad M_t = \frac{\lambda^2}{4} \min \left\{ 1, \; \frac{c_X \, (\lambda |y|)^{\frac{\alpha}{\gamma}}}{\sigma_t^2 \, \|f\|_{\gamma;[0,t]}^{\frac{\alpha}{\gamma}}} \right\}.$$

According to (53) and (54), it is therefore left to prove $\mathbb{E}[M_t^{-p}] < \infty$ for all $p \geq 1$, uniformly in t and y. We trivially have

$$M_t^{-1} \leq \frac{4}{\lambda^2} \max \left\{ 1, \; \frac{\sigma_t^2 \, \|f\|_{\gamma;[0,t]}^{\frac{\alpha}{\gamma}}}{c_X \, (\lambda |y|)^{\frac{\alpha}{\gamma}}} \right\}, \tag{58}$$

and by definition of f in (54)

$$\|f\|_{\gamma;[0,t]} \leq \|J^{-1} V(Z^z)\|_{\gamma;[0,t]} \, |y|.$$

Substituting this value in (58) yields

$$M_t^{-1} \leq \frac{4}{\lambda^2} \max \left\{ 1, \; \frac{\sigma_t^2 \, \|J^{-1} V(Z^z)\|_{\gamma;[0,t]}^{\frac{\alpha}{\gamma}}}{c_X \, \lambda^{\frac{\alpha}{\gamma}}} \right\}. \tag{59}$$

It is thus readily checked that M_t^{-1} admits moments of any order uniformly in t and y, thanks to the fact that $\|J^{-1} V(Z^z)\|_{\gamma;[0,t]}$ admits moments of any order. Indeed, similar arguments as used in [11] to control the p-variation norm of J^{-1} can be used to show that the γ-Hölder norm of J^{-1} admits moments of any order. This concludes the proof for $m = 0$, namely

$$\|\gamma_t^{-1}\|_p \leq c \, \sigma_t^{-2}. \tag{60}$$

Step 2: case $m \geq 1$. Now that we have established (60), the case of higher order derivatives follows from more standard considerations. Indeed, applying elementary rules for the derivative of the inverse to γ_t^{-1}, we get

$$\mathbf{D}(\gamma_t^{-1})^{ij} = -\sum_{k,l=1}^{d} (\gamma_t^{-1})^{ik} (\gamma_t^{-1})^{lj} \mathbf{D}\gamma_t^{kl}. \tag{61}$$

Therefore, it is easily seen that, using the definition of γ_t,

$$\|\mathbf{D}(\gamma_t^{-1})^{ij}\|_{\mathcal{H}} \leq c_d \left(\|\mathbf{D}Z_t\|_{\mathcal{H}} + \|\mathbf{D}^2 Z_t\|_{\mathcal{H}^{\otimes 2}}\right)^2 \|\gamma_t^{-1}\|^2.$$

Together with (47) and (60) this implies

$$\|\mathbf{D}(\gamma_t^{-1})^{ij}\|_{\mathcal{H}} \leq \frac{c_d\, \kappa_t^2}{\sigma_t^4} = \frac{c_d\, \eta_t}{\sigma_t^2},$$

which yields the claim (52) for $m = 1$. Similarly, by using equation (61) repeatedly, we obtain the general case of relation (52). □

We can now conclude this section by giving a short proof of the main theorem.

Proof of Theorem 3.4. We plug the estimates (39), (47) and (52) into (38). This easily yields the claim (35). □

Remark 3.10. Concerning the dependence of the constants c_1, c_2 in (35) on T we note the following: *(i)* An analysis of the proof of Proposition 3.7 yields that c_2 can be chosen independently of the time horizon T.

(ii) The dependence of c_1 on T is less explicit, since it relies on the constant c_X appearing in Hypothesis (2.21), which in turn is intimately linked to the variance of the driving process X (cf. e.g. Example 5.4). In the case of fractional Brownian motion, Hardy-Littlewood's lemma (see e.g [27, Equation (5.20)]) reveals that c_X is bounded from below uniformly in T. Assuming that this is the case, an analysis of the derivation of (47) shows that c_2 depends on T via $M^{\kappa_T^{2/(1+1/\rho)}}$ for some $M > 1$.

4. Varadhan estimate

Fix a small parameter $\varepsilon \in (0, 1]$, and consider the solution Z_t^ε to the stochastic differential equation

$$Z_t^\varepsilon = z + \int_0^t V_0(Z_s^\varepsilon) ds + \varepsilon \sum_{i=1}^{d} \int_0^t V_i(Z_s^\varepsilon) dX_s^i, \quad \forall t \in [0, T], \tag{62}$$

where, as before, the vector fields $V_0, V_1, \ldots, V_d$ are C^∞-bounded vector fields on $\mathbb{R}^n$. In this section we will work under the same assumptions as in Section 3 which are summarized as follows.

Hypothesis 4.1. *Let X be an $\mathbb{R}^d$-valued continuous, centered Gaussian process starting at zero with i.i.d. components and covariance function R satisfying Hypothesis 2.11. We further assume that X satisfies Hypothesis 2.18 and 2.21 and that the vector fields $V_1, \ldots, V_d$ satisfy Hypothesis 3.1. Without loss of generality we choose $T = 1$.*

With Hypothesis 4.1 at hand, we will describe the asymptotic behavior of the density of Z_t^ε as $\varepsilon \to 0$. We start by recalling the large deviation setting for rough paths in Section 4.1, and will complete the estimates in Section 4.2.

4.1. **Large deviations setting.** Let us first recall that under Hypothesis 4.1, X can be lifted to a p-rough path with $p > 2\rho$. According to the general rough path theory (see, e.g., inequality (10.15) and Theorem 15.33 in [17]), for any positive λ and $\delta < 2/p$ we have

$$\mathbb{E}\left[\exp\left(\lambda \sup_{t\in[0,1],\epsilon\in(0,1]} |Z_t^\varepsilon|^\delta\right)\right] < \infty. \tag{63}$$

In addition, the Malliavin derivative and Malliavin matrix of Z_1^ε can be controlled using the same arguments as in the previous section. More precisely, replacing the V_i's with εV_i's in the proof of Propositions 3.8 and 3.9, we have

$$\sup_{\varepsilon\in(0,1]} \|Z_1^\varepsilon\|_{k,r} < \infty, \qquad \text{for each } k \geq 1 \text{ and } r \geq 1; \tag{64}$$

$$\|\gamma_{Z_1^\varepsilon}^{-1}\|_r \leq c_r \varepsilon^{-2}, \quad \text{for any } r \geq 1, \tag{65}$$

where $\gamma_{Z_1^\varepsilon}$ is the Malliavin matrix of Z_1^ε.

Denote by $\mathbf{J}^\varepsilon$ the Jacobian of Z^ε. Similar to (30), the process $\mathbf{J}^\varepsilon$ is the unique solution to the linear equation

$$\mathbf{J}_t^\varepsilon = \mathrm{Id}_n + \int_0^t DV_0(Z_s^\varepsilon)\mathbf{J}_s^\varepsilon ds + \varepsilon \sum_{j=1}^d \int_0^t DV_j(Z_s^\varepsilon)\, \mathbf{J}_s^\varepsilon \, dX_s^j.$$

Its moments are uniformly bounded (in $\varepsilon \in (0,1]$) in the next proposition.

Proposition 4.2. *For any $\eta \geq 1$, there exists a finite constant c_η such that the Jacobian $\mathbf{J}^\varepsilon$ satisfies*

$$\sup_{\varepsilon\in(0,1]} \mathbb{E}\left[\|\mathbf{J}^\varepsilon\|_{p-\mathrm{var};[0,1]}^\eta\right] = c_\eta. \tag{66}$$

Proof. When $\varepsilon = 1$, the integrability of $\mathbf{J}^\varepsilon$ is proved in [11], and has been recalled in Proposition 2.27 above. It can be checked that the estimates in [11] only depends on the supremum norm of the vector fields and their derivatives. In the present case, the vector fields εV_i in equation (62) are uniformly bounded in $\varepsilon \in (0,1]$ together with their derivatives. Hence the uniform integrability of $\mathbf{J}^\varepsilon$ (in ε) follows. □

In order to state a large deviation type result, let us introduce the so-called skeleton of equation (62), that is, we introduce the map $\Phi : \bar{\mathcal{H}} \to \mathcal{C}([0,1], \mathbb{R}^n)$ associating to each $h \in \bar{\mathcal{H}}$ the unique solution of the ordinary differential equation

$$\Phi_t(h) = z + \int_0^t V_0(\Phi_s(h))ds + \sum_{i=1}^d \int_0^t V_i(\Phi_s(h))dh_s^i. \tag{67}$$

By the embedding Theorem 2.16, for each $h \in \bar{\mathcal{H}}$, the above equation can be understood in Young sense. In particular, it follows that there is a unique solution $\Phi_\cdot(h)$. Moreover, Φ_t is a differentiable mapping from $\bar{\mathcal{H}}$ to the space $\mathcal{C}([0,1],\mathbb{R}^n)$. We let $\gamma_{\Phi_1(h)}$ be the deterministic Malliavin matrix of $\Phi_1(h)$, that is,

$$\gamma^{ij}_{\Phi_1(h)} = \langle \mathbf{D}\Phi^i_1(h), \mathbf{D}\Phi^j_1(h)\rangle_{\mathcal{H}}. \tag{68}$$

Along the same lines, we introduce the Jacobian $J(h)$ of equation (67), that is the unique solution of the following equation

$$J_t(h) = \mathrm{Id}_n + \sum_i \int_0^t DV_i(\Phi_s(h))J_s(h)dh^i_s + \int_0^t DV_0(\Phi_s(h))J_s(h)ds. \tag{69}$$

Remark 4.3. For a geometric p-rough path $\mathbf{x}$, it is sometimes convenient to write $\Phi(\mathbf{x})$ obtained by solving (67) with h replaced with $\mathbf{x}$. By general theory of rough path, Φ is a continuous function of $\mathbf{x}$ in the p-variation topology. We will use this notation without further mention when there is no confusion.

Remark 4.4. Let X be an $\mathbb{R}^d$-valued Gaussian process satisfying Hypothesis 4.1 and let $h \in \bar{\mathcal{H}}$ be an element of the Cameron-Martin space of X. We use the notation $\mathbf{X}+h$ to denote lift of $X+h$ to a p-rough path. This construction is made possible by the embedding in Theorem 2.16 and Young's pairing. We direct the readers to Section 9.4 of [17] for more details.

We next note that, following the same arguments as in [7], for each $h \in \bar{\mathcal{H}}$,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\left(\Phi_t(\varepsilon\mathbf{X}+h) - \Phi_t(h)\right) = G_t(h), \tag{70}$$

in the topology of $\mathbb{D}^\infty$ for some random variable $G_t(h)$. The equation satisfied by $G_t(h)$ is obtained by formally differentiating (67) with respect to ε, which yields

$$\begin{aligned} G_t(h) = \sum_i \int_0^t DV_i(\Phi_s(h))G_s(h)dh^i_s + \int_0^t DV_0(\Phi_s(h))G_s(h)ds \\ + \sum_i \int_0^t V_i(\Phi_s(h))dX^i_s. \end{aligned} \tag{71}$$

Comparing equations (71) and (69), an elementary variational principle argument reveals that

$$G_t(h) = J_t(h)\int_0^t (J_s(h))^{-1}V_i(\Phi_s(h))dX^i_s, \tag{72}$$

which implies that $G_t(h)$ is a centered Gaussian random variable. Moreover, starting from equation (72), some easy computations show that the Malliavin derivative of $G_t(h)$ and the deterministic Malliavin derivative of Φ at h coincide. Hence, the covariance matrix of $G_1(h)$ is the deterministic Malliavin matrix $\gamma_{\Phi_1(h)}$.

As a last preliminary step we recall the large deviation principle for stochastic differential equations driven by Gaussian rough path, which is the basis for Varadhan type estimates and is standard in rough paths theory (see [17, Section 19.4]).

Theorem 4.5. *Let Φ be as in (67), Z_1^ε be the solution to equation (62) and set*

$$I(y) := \inf_{\Phi_1(h)=y} \frac{1}{2}\|h\|^2_{\bar{\mathcal{H}}} \quad \forall y \in \mathbb{R}^n.$$

Then Z_1^ε satisfies a large deviation principle with rate function $I(y)$.

Proof. First, it is known (see, e.g., [17, Theorem 15.55]) that $\varepsilon\mathbf{X}$, as a p-rough path, satisfies a large deviation principle in the p-variation topology with good rate function given by

$$J(h) = \begin{cases} \frac{1}{2}\|h\|^2_{\bar{\mathcal{H}}} & \text{if } h \in \bar{\mathcal{H}} \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, by Remark 4.3, $\Phi_1(\mathbf{x})$ is continuous function of $\mathbf{x}$ in p-variation topology. Since $Z_1^\varepsilon = \Phi_1(\varepsilon\mathbf{X})$ the result follows from the contraction principle. □

4.2. **Asymptotic behavior of the density.** Recall that the skeleton Φ is defined by (67). Our density estimates will involve a "distance" which depends on Φ as follows

$$d^2(y) = I(y) = \inf_{\Phi_1(h)=y} \frac{1}{2}\|h\|^2_{\bar{\mathcal{H}}}, \quad \text{and} \quad d_R^2(y) = \inf_{\Phi_1(h)=y, \det \gamma_{\Phi_1(h)}>0} \frac{1}{2}\|h\|^2_{\bar{\mathcal{H}}}. \tag{73}$$

Interestingly enough, the two distances defined above coincide under the ellipticity assumptions.

Lemma 4.6. *Assume that Hypothesis 4.1 is satisfied. Then we have $d^2(y) = d_R^2(y)$ for every $y \in \mathbb{R}^n$.*

Proof. The claimed identity is mainly due to the uniform ellipticity of the vector fields $V_i's$. Indeed, pick any $h \in \bar{\mathcal{H}}$ such that $\Phi_1(h) = y$. Recall that $J(h)$ is the Jacobian of the deterministic equation (67) and $\gamma_{\Phi_1(h)}$ is the deterministic Malliavin matrix of Φ at h. Similarly to (32) we have

$$\mathbf{D}_s^k \Phi_1(h) = J_1(h)(J_s(h))^{-1} V_k(\Phi_s(h)).$$

Therefore, owing to the definition (68) of the Malliavin matrix, we get the following identity for all $x \in \mathbb{R}^n$

$$\begin{aligned} \sum_{ij} x_i \gamma^{ij}_{\Phi_1(h)} x_j &= \sum_k \Big\| \sum_i x_i (\mathbf{D}^k \Phi_1(h))^i \Big\|^2_{\mathcal{H}} \\ &= \int_0^t \int_0^t \big\langle x^T J_{u1}(h) V(\Phi_u(h))\,,\, x^T J_{v1}(h) V(\Phi_v(h)\big\rangle\, dR(u,v). \end{aligned}$$

Let us now define a function f by

$$f_u = x^T J_{u1}(h) V(\Phi_u(h)).$$

Under the same assumptions as in Proposition 2.23, which are satisfied due to Hypothesis 4.1, we have the interpolation inequality (see relation (27))

$$\int_0^1 \int_0^1 \langle f_u, f_v \rangle dR(u,v) \geq \frac{1}{4}\sigma_1^2 \|f\|^2_{\infty;[0,1]} \min\left\{1, \frac{2\left(\frac{c_X}{2}\right)^{\frac{2\gamma+\alpha}{4\gamma}}}{\sigma_1} \frac{\|f\|^{\frac{\alpha}{2\gamma}}_{\infty;[0,1]}}{(1+\|f\|^{\frac{\alpha}{2\gamma}}_{\gamma;[0,1]})}\right\}^2.$$

Furthermore, the uniform ellipticity condition implies that for any $x \neq 0$,

$$\|f\|_{\infty;[0,1]} > 0.$$

Therefore, the deterministic Malliavin matrix $\gamma_{\Phi_1(h)}$ is non-degenerate at h. In conclusion, for any $h \in \bar{\mathcal{H}}$ such that $\Phi_1(h) = y$ we have $\det \gamma_{\Phi_1(h)} > 0$ and thus $d_R(y) \equiv d(y)$. □

Now we can state the main result of this section, giving the logarithmic asymptotic behavior of the density as $\varepsilon \to 0$.

Theorem 4.7. *Let Z^ε be the process defined by (62), and denote by $p_\varepsilon(y)$ the density of Z_1^ε. Due to Hypothesis 4.1, we have*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \log p_\varepsilon(y) = -d^2(y),$$

where d is the function defined by (73).

Proof. With the previous estimates in hand, the proof is similar to the one of [7, Theorem 3.2]. For the reader's convenience, we give some details below. Let us divide the proof in two steps.

Step 1: Lower bound. We shall prove that

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^2 \log p_\varepsilon(y) \geq -d_R^2(y). \tag{74}$$

To this aim, fix $y \in \mathbb{R}^n$. We only need to show (74) for $d_R^2(y) < \infty$, since the statement is trivial whenever $d_R^2(y) = \infty$. Next fix an arbitrary $\eta > 0$ and let $h \in \bar{\mathcal{H}}$ be such that $\Phi_1(h) = y$ and $\|h\|_{\bar{\mathcal{H}}}^2 \leq d_R^2(y) + \eta$. Let $f \in C_0^\infty(\mathbb{R}^n)$. By Cameron-Martin's theorem for the Gaussian process X, it is readily checked that

$$\mathbb{E}\left[f(Z_1^\varepsilon)\right] = e^{-\frac{\|h\|_{\bar{\mathcal{H}}}^2}{2\varepsilon^2}}\, \mathbb{E}\left[f(\Phi_1(\varepsilon X + h)) e^{-\frac{X(h)}{\varepsilon}}\right],$$

where $X(h)$ denotes the Wiener integral of h with respect to X introduced in Section 2.2. We now proceed by means of a truncation argument: consider a function $\chi \in C^\infty(\mathbb{R})$, satisfying $0 \leq \chi \leq 1$, such that $\chi(t) = 0$ if $t \notin [-2\eta, 2\eta]$, and $\chi(t) = 1$ if $t \in [-\eta, \eta]$. Then, if $f \geq 0$, we have

$$\mathbb{E}\left[f(Z_1^\varepsilon)\right] \geq e^{-\frac{\|h\|_{\bar{\mathcal{H}}}+4\eta}{2\varepsilon^2}}\, \mathbb{E}\left[\chi(\varepsilon X(h)) f(\Phi_1(\varepsilon X + h))\right].$$

Hence, by means of an approximation argument applying the above estimate to $f = \delta_y$, we obtain

$$\varepsilon^2 \log p_\varepsilon(y) \geq -\left(\frac{1}{2}\|h\|_{\bar{\mathcal{H}}}^2 + 2\eta\right) + \varepsilon^2 \log \mathbb{E}\big[\chi(\varepsilon X(h)) \delta_y(\Phi_1(\varepsilon X + h))\big]. \tag{75}$$

Indeed, for any non-degenerate random vector F, the distribution on Wiener's space $\delta_y(F)$ is an element in $\mathbb{D}^{-\infty}$, the dual of $\mathbb{D}^\infty$. The expression $\mathbb{E}[\delta_y(F)G]$ can thus be interpreted as the coupling $\langle \delta_y(F), G\rangle$ for any $G \in \mathbb{D}^\infty$ (see [27, Section 2.1.5]).

Let us now bound the right hand side of equation (75). Owing to the fact that $\Phi_1(h) = y$ and thanks to the scaling properties of the Dirac distribution, it is easily seen that

$$\mathbb{E}\big(\chi(\varepsilon X(h)) \delta_y(\Phi_1(\varepsilon X + h))\big) = \varepsilon^{-n} \mathbb{E}\left(\chi(\varepsilon X(h)) \delta_0\left(\frac{\Phi_1(\varepsilon X + h) - \Phi_1(h)}{\varepsilon}\right)\right).$$

In addition, according to the definition (70), we have

$$\lim_{\varepsilon\downarrow 0}\frac{\Phi_1(\varepsilon X+h)-\Phi_1(h)}{\varepsilon}=G_1(h),$$

and recall that we have established, thanks to (72), that $G_1(h)$ is an n-dimensional random vector in the first Wiener chaos with variance $\gamma_{\Phi_1(h)}>0$. Hence, $G_1(h)$ is non-degenerate and integrating by parts combined with standard arguments from Malliavin calculus yields

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\chi(\varepsilon X(h))\delta_0\left(\frac{\Phi_1(\varepsilon X+h)-\Phi_1(h)}{\varepsilon}\right)\right]=\mathbb{E}\left[\delta_0(G_1(h))\right]. \tag{76}$$

In particular, we get

$$\lim_{\varepsilon\downarrow 0}\varepsilon^2\log\mathbb{E}\big(\chi(\varepsilon X(h))\delta_y(\Phi_1(\varepsilon X+h))\big)=0.$$

Plugging this information in (75) and letting $\varepsilon\downarrow 0$ we end up with

$$\liminf_{\varepsilon\downarrow 0}\varepsilon^2\log p_\varepsilon(y)\geq-\left(\frac{1}{2}\|h\|_{\mathcal{H}}^2+2\eta\right)\geq-\left(d_R^2(y)+3\eta\right).$$

Since $\eta>0$ is arbitrary this yields (74). At this point we can notice that we have chosen h such that $\|h\|_{\mathcal{H}}^2\leq d_R^2(y)+\eta$ in order to get a non degenerate random variable $G_1(h)$ in (76).

Step 2: Upper bound. Next, we show that

$$\limsup_{\varepsilon\downarrow 0}\varepsilon^2\log p_\varepsilon(y)\leq -d^2(y). \tag{77}$$

Towards this aim, fix a point $y\in\mathbb{R}^n$ and consider a function $\chi\in C_0^\infty(\mathbb{R}^n), 0\leq\chi\leq 1$ such that χ is equal to one in a neighborhood of y. The density of Z_1^ε at point y is given by

$$p_\varepsilon(y)=\mathbb{E}\left[\chi(Z_1^\varepsilon)\delta_y(Z_1^\varepsilon)\right].$$

Next integrate the above expression by parts in the sense of Malliavin calculus thanks to Proposition 3.5. This yields

$$\begin{aligned}\mathbb{E}[\chi(Z_1^\varepsilon)\delta_y(Z_1^\varepsilon)]=&\mathbb{E}\left[\mathbf{1}_{\{Z_1^\varepsilon>y\}}H_{(1,2,\dots,n)}(Z_1^\varepsilon,\chi(Z_1^\varepsilon))\right]\\ \leq&\mathbb{E}\left[|H_{(1,2,\dots,n)}(Z_1^\varepsilon,\chi(Z_1^\varepsilon))|\right]\\ =&\mathbb{E}\big[|H_{(1,2,\dots,n)}(Z_1^\varepsilon,\chi(Z_1^\varepsilon))|\mathbf{1}_{\{Z_1^\varepsilon\in\mathrm{supp}\chi\}}\big]\\ \leq&\mathbb{P}(Z_1^\varepsilon\in\mathrm{supp}\chi)^{\frac{1}{q}}\|H_{(1,..,n)}(Z_1^\varepsilon,\chi(Z_1^\varepsilon))\|_p,\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Furthermore, relation (37) and an application of Hölder's inequality (see, e.g., [27, Proposition 1.5.6]) gives

$$\|H_{(1,\dots,n)}(Z_1^\varepsilon,\chi(Z_1^\varepsilon))\|_p\leq c_{p,q}\|\gamma_{Z_1^\varepsilon}^{-1}\|_\beta^m\|\mathbf{D}Z_1^\varepsilon\|_{n,\gamma}^r\|\chi(Z_1^\varepsilon)\|_{n,q}^n,$$

for some constants $\beta,\gamma>0$ and integers m,r. Thus, invoking the estimates (64) and (65), we obtain

$$\lim_{\varepsilon\downarrow 0}\varepsilon^2\log\|H_{(1,\dots,n)}(Z_1^\varepsilon,\chi(Z_1^\varepsilon))\|_p=0.$$

Finally the large deviation principle for Z_1^ε recalled in Theorem 4.5 ensures that for small ε we have

$$\mathbb{P}(Z_1^\varepsilon\in\mathrm{supp}\chi)^{\frac{1}{q}}\leq e^{-\frac{1}{q\varepsilon^2}(\inf_{z\in\mathrm{supp}\chi}d^2(z)+o(1))}.$$

Since q can be chosen arbitrarily close to 1 and $\operatorname{supp}(\chi)$ can be taken arbitrarily close to y, the proof of (77) is now easily concluded thanks to the lower semi-continuity of d.

Combining Lemma 4.6, (74) and (77), the proof of Theorem 4.7 is thus completed. □

5. Applications

Our main results, Theorem 3.4 and Theorem 4.7 rely on Hypothesis 2.11, 2.18 and 2.21. Let us also recall that the density bound (35) involves a coefficient η defined by (34). In this section we provide explicit examples of Gaussian processes satisfying the aforementioned assumptions and give estimates for η as a function of t.

Remark 5.1. The interpolation inequalities in Proposition 2.19 and Proposition 2.23 rely on an integral representation for the Cameron-Martin norm related to X (see relation (12)), which is satisfied for Gaussian processes starting at zero. We note that this is not a restriction in applications, since the RDE (2) driven by X is the same as the one driven by $\tilde{X} = \{\tilde{X}_t = X_t - X_0, t \geq 0\}$. Moreover, one easily checks that if X satisfies Hypotheses 2.11, 2.18 and 2.21, then so does $\tilde{X}$.

Remark 5.2. Suppose that X_t is a continuous, centered real-valued Gaussian processes with covariance R. Then

(i) If $\partial^2_{ab} R \leq 0$ in the sense of distributions, then Hypothesis 2.18, (i) is satisfied.
(ii) If $\sigma^2_{s,t} = F(|t-s|)$ for some continuous, non-decreasing function F then Hypothesis 2.18, (ii) is satisfied.
(iii) If X starts at zero, satisfies Hypothesis 2.18, (i) and $\partial_a R(a,b) \geq 0$ for $a < b$ in the sense of distributions, then Hypothesis 2.18, (ii) is satisfied.

Proof. We first note that (i) is proved in [12, Lemma 2.20] and (iii) follows from [10, Section 4.2.1]. For (ii): We have

$$\begin{aligned} 2R^{st}_{uv} &= \sigma^2_{s,v} - \sigma^2_{s,u} + \sigma^2_{u,t} - \sigma^2_{v,t} \\ &= F(|v-s|) - F(|u-s|) + F(|t-u|) - F(|t-v|). \end{aligned}$$

Since F is non-decreasing this implies, for $s \leq u \leq v \leq t$, $2R^{st}_{uv} \geq 0$. □

With this remark in mind, we are now ready to provide a series of examples to which the results of Sections 3 and 4 apply.

Example 5.3. Let B^H be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. As mentioned in Remark 3.3, in this case η_t does not depend on t due to the self-similarity of B^H. It is also shown in [10] that Hypothesis 2.18 and 2.21 are satisfied whenever $H \in (\frac{1}{4}, \frac{1}{2})$. In [12, Example 2.8] it is proved that B^H has Hölder-controlled mixed $(1,\rho)$-variation and thus Hypothesis 2.11 is satisfied.

Example 5.4. Let X be a d-dimensional centred Gaussian process with i.i.d. components, such that the coefficient $\sigma^2_{s,t}$ defined by (7) satisfies the following relation

$$\sigma^2_{s,t} = F\big(|t-s|\big) \geq 0,$$

for some non-negative, concave function F satisfying $F(0) = 0$ and

$$\inf_{s\in[0,T]} F'(s) > 0. \tag{78}$$

We note that if F is not identically equal to zero, then $F(0) = 0$, $F \geq 0$ and concavity imply that (78) is satisfied for some $T > 0$. In addition, we assume that

$$C_1 t^{\frac{1}{\rho}} \leq F(t) \leq C_2 t^{\frac{1}{\rho}} \quad \forall t \in [0,T], \tag{79}$$

for some $\rho \in [1,2)$, $C_1, C_2 > 0$. Since $2R(s,t) = -F(|t-s|) + F(t) + F(s)$, concavity of F and the fact that F is increasing imply Hypothesis 2.18, due to Remark 5.2. It is readily checked from [12, Example 2.9] that under assumption (79) we have

$$V_{1,\rho}\big(R; [s,t]^2\big) \leq C|t-s|^{1/\rho}$$

for some constant $C > 0$ and thus X has Hölder-controlled mixed $(1,\rho)$-variation. Recalling that $\sigma_t^2 := \sigma_{0,t}^2$, invoking (79) again we obtain

$$\eta_t = \frac{V_{1,\rho}\big(R; [0,t]^2\big)}{\sigma_t^2} \leq C.$$

In particular, η is bounded on $[0,T]$. Finally, from [12, Theorem 6.1] we have that Hypothesis 2.21 is satisfied with $\alpha = 1$.

Example 5.5. Let $X = B^{H_1} + B^{H_2}$ be a sum of two independent fBm with Hurst parameters $H_1, H_2 \leq 1/2$. Then

$$\sigma_{s,t}^2 = |t-s|^{2H_1} + |t-s|^{2H_2} =: F\big(|t-s|\big)$$

and the previous example applies.

Example 5.6. Consider a *bifractional Brownian motion* (cf., e.g., [20, 31, 23]), that is, a centered Gaussian process $B^{H,K}$ on $[0,T]$ with covariance function given by[1]

$$R(s,t) = \frac{1}{2^K}\Big(\big(s^{2H} + t^{2H}\big)^K - |t-s|^{2HK}\Big),$$

for some $H \in (0,1)$ and $K \in (0,1]$ such that $HK \leq 1/2$. Since $B^{H,K}$ is a self-similar process with index HK, the coefficient η does not depend on t. Hypothesis 2.18 and the fact that R admits a Hölder-controlled mixed $(1,\rho)$-variation, i.e. Hypothesis 5.2, have been verified in [12, Example 2.12]. In order to check Hypothesis 2.21 we recall from [12, equation (6.2)], using Hypothesis 2.18, that

$$2\mathrm{Var}(X_{s,t}|\mathcal{F}_{0,s} \vee \mathcal{F}_{t,T}) \geq 2R\begin{pmatrix} 0 & T \\ s & t \end{pmatrix}.$$

Hence,

$$\begin{aligned}
2\mathrm{Var}(X_{s,t}|\mathcal{F}_{0,s} \vee \mathcal{F}_{t,T}) &\geq 2\mathbb{E}(X_T - X_0)(X_t - X_s) = 2(R(T,t) - R(T,s)) \\
&= 2^{1-K}\big((t^{2H} + T^{2H})^K - |t-T|^{2HK}\big) - \big((s^{2H} + T^{2H})^K - |s-T|^{2HK}\big) \\
&\geq 2^{1-K}\big(|s-T|^{2HK} - |t-T|^{2HK}\big) \\
&\geq C(T)|t-s|,
\end{aligned}$$

which implies Hypothesis 2.21.

[1] As pointed out, for example, in [23] this process does not fit in the Volterra framework.

Example 5.7. Consider a random Fourier series[2]

$$\Psi(t) = \sum_{k=1}^{\infty} \alpha_k Y^k \sin(kt) + \alpha_{-k} Y^{-k} \cos(kt), \qquad t \in [0, 2\pi],$$

with zero-mean, independent Gaussians $\{Y^k;\ k \in \mathbb{Z}\}$ with unit variance. Then the covariance R can be computed in an elementary way

$$\begin{aligned} R(s,t) &= \sum_{k=1}^{\infty} \alpha_k^2 \sin(ks)\sin(kt) + \alpha_{-k}^2 \cos(ks)\cos(kt) \\ &= \frac{1}{2}\sum_{k=1}^{\infty} (\alpha_k^2 + \alpha_{-k}^2)\cos(k(t-s)) + (\alpha_k^2 - \alpha_{-k}^2)\cos(k(t+s)). \end{aligned} \tag{80}$$

Let us consider the special case where Ψ is a stationary random field. This implies $\alpha_k^2 = \alpha_{-k}^2$ and thus

$$R(s,t) = K(|t-s|), \quad \text{and} \quad \sigma_{s,t}^2 = 2(K(0) - K(|t-s|)) =: F(|t-s|),$$

where the function K is defined by

$$K(t) := \sum_{k=1}^{\infty} \alpha_k^2 \cos(kt).$$

We now wish to prove that this situation can be seen as a particular case of Example 5.4. For simplicity we concentrate on the model-case

$$\alpha_k^2 = C k^{-(1+\frac{1}{\rho})}. \tag{81}$$

for some $\rho \in [1,2)$, $C > 0$. For more general conditions on the coefficients we refer to [12, Section 3]. By [12, Section 3], K is convex on $[0, 2\pi]$, decreasing on $[0, \pi]$ and $\frac{1}{\rho}$-Hölder continuous. In order to check the conditions of Example 5.4, it remains to verify the lower bound in (79). We observe

$$\begin{aligned} F(t) = K(0) - K(t) = \sum_{k=1}^{\infty} \alpha_k^2 (1 - \cos(kt)) = 2\sum_{k=1}^{\infty} \alpha_k^2 \sin^2\!\left(\frac{kt}{2}\right) \geq 2 \sum_{k=\lfloor \frac{1}{2t} \rfloor}^{\lfloor \frac{1}{t} \rfloor} \alpha_k^2 \sin^2\!\left(\frac{kt}{2}\right) \\ \gtrsim \sum_{k=\lfloor \frac{1}{2t} \rfloor}^{\lfloor \frac{1}{t} \rfloor} \alpha_k^2 \gtrsim \alpha_{\lfloor \frac{1}{t} \rfloor}^2 \left(\lfloor \frac{1}{t} \rfloor - \lfloor \frac{1}{2t} \rfloor\right) \gtrsim \alpha_{\lfloor \frac{1}{t} \rfloor}^2 \lfloor \frac{1}{t} \rfloor \gtrsim t^{\frac{1}{\rho}}, \end{aligned}$$

where we write $a \gtrsim b$ whenever $a \geq c\, b$ for a universal constant c and where we have used inequality (81) for the last step. Since F is not identically equal to zero, it follows that there is a time $T \in (0, 2\pi]$, such that F is concave, $\inf_{s \in [0,T]} F'_-(s) > 0$, F is $\frac{1}{\rho}$-Hölder continuous and (79) is satisfied. Hence, by Example 5.4 Hypothesis 2.11, 2.18 and 2.21 are satisfied and η is bounded on $[0, T]$.

[2]We may ignore the (constant, random) zero-mode in the series since we are only interested in properties of the increments of the process.

Example 5.8. Let X be a d-dimensional continuous, centred Gaussian process with i.i.d. components. In the following X_t denotes one of its components. Assume that X_t is a stationary, zero-mean process with covariance

$$R(s,t) = K\big(|t-s|\big)$$

for some continuous and positive definite function K. By Bochner's Theorem there is a finite positive symmetric measure μ on $\mathbb{R}$ such that

$$K(t) = \int \cos(t\xi)\mu(d\xi)$$

and thus

$$\sigma^2(t) := \sigma^2_{0,t} = 2\big(K(0) - K(t)\big) = 4\int \sin^2(t\xi/2)\mu(d\xi).$$

The case of discrete μ corresponds to Example 5.7. Another example is given by the *fractional Ornstein–Uhlenbeck process*,

$$X_t = \int_{-\infty}^{t} e^{-\lambda(t-u)}\, dB^H_u, \qquad t \in \mathbb{R}.$$

In this case, it is known that X has a *spectral density* $\mu(d\xi)$ such that

$$\frac{d\mu}{d\xi} = c_H \frac{|\xi|^{1-2H}}{\lambda^2+\xi^2} \equiv \hat{K}(\xi). \tag{82}$$

By Theorem 7.3.1 in [26] we have that if $\hat{K}$ is regularly varying at ∞, then the coefficient σ_t defined by (7) satisfies $\sigma_t^2 \sim \frac{C\hat{K}(1/t)}{t}$ as $t \to 0$ which in the case of (82) implies that there exists a $T > 0$ such that

$$C_1 t^{2H} \le \sigma^2(t) \le C_2 t^{2H} \quad \text{for all } t \in [0,T].$$

Moreover, it can be seen that there is a $T > 0$ such that K is convex on the interval $[0,T]$ (cf. [12, Example 5.3]) and $\sup_{t\in[0,T]} K'(t) < 0$. Hence, Hypothesis 2.18 and by [12, equation (6.2)] Hypothesis 2.21 are satisfied. By [12] we conclude

$$V_{1,\rho}(R;[s,t]^2) = O(|t-s|^{2H}) \text{ for all } [s,t] \subseteq [0,T].$$

Hence, Hypothesis 2.11 is satisfied and

$$\eta_t \le C \quad \text{for all } t \in [0,T].$$

References

[1] V. Bally, E. Pardoux: Malliavin calculus for white noise driven parabolic SPDEs. *Potential Anal.* **9** (1998), no. 1, 27–64.

[2] F. Baudoin, M. Hairer: A version of Hörmander's theorem for the fractional Brownian motion. *Probab. Theory Related Fields* **139** (2007), no. 3-4, 373–395.

[3] F. Baudoin, E. Nualart, C. Ouyang, S. Tindel: On probability laws of solutions to differential systems driven by fractional Brownian motion. *Ann. Probab.*, **44** (2016), no. 4, 2554-25901.

[4] F. Baudoin, C. Ouyang: Small-time kernel expansion for solutions of stochastic differential equations driven by fractional Brownian motions. *Stoch. Proc. Appl.* **121** (2011), no. 4, 759–792.

[5] F. Baudoin, C. Ouyang, S. Tindel: Upper bounds for the density of solutions of stochastic differential equations driven by fractional Brownian motions. *Ann. Inst. Henri Poincaré Probab. Stat.*, **50** (1), 2014, 111-135.

[6] D. Bell, S.-E. Mohammed: The Malliavin calculus and stochastic delay equations. *J. Funct. Anal.* **99** (1991), no. 1, 75–99.
[7] F. Baudoin, C. Ouyang, X. Zhang: Varadhan estimates for RDEs driven by fractional Brownian motions. *Stochastic Processes and Their Applications*, Vol. 125, Issue 2, 634-652, (2015)
[8] T. Cass, P. Friz: Densities for rough differential equations under Hörmander's condition. *Ann. Math.* **171**, (2010), 2115–2141.
[9] T. Cass, P. Friz, N. Victoir: Non-degeneracy of Wiener functionals arising from rough differential equations. *Trans. Amer. Math. Soc.* **361** (2009), no. 6, 3359–3371.
[10] T. Cass, M. Hairer, C. Litterer, S. Tindel: Smoothness of the density for solutions to Gaussian Rough Differential Equations. *Ann. Probab.*, **43** (2015), no. 1, 188-239.
[11] T. Cass, C. Litterer, T. Lyons: Integrability and tail estimates for Gaussian rough differential equations. *Ann. Probab* **41** (2013), no. 4, 3026–3050.
[12] P.K. Friz, B. Gess, A. Gulisashvili, S. Riedel: Jain-Monrad criterion for rough paths. *Ann. Probab.*, *44* (1), 684-738 (2016).
[13] P.K. Friz, S. Riedel: Integrability of (Non-)Linear Rough Differential Equations and Integrals. *Stochastic Analysis and Applications*, Vol 31, Issue 2, (2013).
[14] M. Ferrante, C. Rovira, M. Sanz-Solé: Stochastic delay equations with hereditary drift: estimates of the density. *J. Funct. Anal.* **177** (2000), no. 1, 138–177.
[15] P.K. Friz, M. Hairer: *A course on rough paths: with an introduction to regularity structures,* Universitext, Springer (2014).
[16] P.K. Friz, N. Victoir: Differential equations driven by Gaussian signals. Ann. Inst. Henri Poincaré Probab. Stat. 46 (2010), no. 2, 369–413.
[17] P.K. Friz, N. Victoir: *Multidimensional Stochastic Processes as Rough Paths.* Cambridge University Press (2010).
[18] M. Gubinelli: Controlling rough paths. *J. Funct. Anal.* **216**, 86-140 (2004).
[19] M. Hairer: Ergodicity of stochastic differential equations driven by fractional Brownian motion. Ann. Probab., **33**, No. 2 (2005), 703-758.
[20] C. Houdré, J. Villa: *An example of infinite dimensional quasi-helix*, Stochastic models (Mexico City, 2002), Contemp. Math., vol. 336, Amer. Math. Soc., Providence, RI, 2003, pp. 195–201.
[21] Y. Inahama: Malliavin differentiability of solutions of rough differential equations. *J. Funct. Anal.*, **267** (5), 1566-1584 (2013).
[22] Y. Inahama: Short time kernel asymptotics for rough differential equation driven by fractional Brownian motion. Eletron. J. Probab., **21** (2016), no. 34, 1-29.
[23] I. Kruk, F. Russo, C.A. Tudor: *Wiener integrals, Malliavin calculus and covariance measure structure*, J. Funct. Anal. **249** (2007), no. 1, 92–142.
[24] S. Lou and C. Ouyang: Local times of stochastic differential equations driven by fractional Brownian motions. *Stochastic Process Appl.*, Vol. 127, Issue 11, 3643-3660, (2017).
[25] P. Malliavin: Stochastic calculus of variation and hypoelliptic operators. Proceedings of the International Symposium on Stochastic Differential Equations, pp. 195–263. Wiley, 1978.
[26] M.B. Marcus, J. Rosen: *Markov processes, Gaussian processes, and local times*, Cambridge Studies in Advanced Mathematics, vol. 100, Cambridge University Press, Cambridge, 2006.
[27] D. Nualart: *The Malliavin Calculus and Related Topics.* Probability and its Applications. Springer-Verlag, 2nd Edition, (2006).
[28] D. Nualart, L. Quer-Sardanyons: Existence and smoothness of the density for spatially homogeneous SPDEs. *Potential Anal.* **27** (2007), no. 3, 281–299.
[29] D. Nualart, B. Saussereau: Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion. *Stochastic Process. Appl* **119** (2009), no. 2, 391–409.
[30] C. Rovira, M. Sanz-Solé: The law of the solution to a nonlinear hyperbolic SPDE. *J. Theoret. Probab.* **9** (1996), no. 4, 863–901.
[31] F. Russo, C.A. Tudor: *On bifractional Brownian motion*, Stochastic Process. Appl. **116** (2006), no. 5, 830–856.
[32] N. Towghi: Multidimensional extension of L. C. Young's inequality, *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 22, 13 pp. (electronic).

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